

# How Fitness Aggregation Methods Affect the Performance of Competitive CoEAs on Bilinear Problems

Mario Alejandro Hevia Fajardo  
School of Computer Science  
University of Birmingham, Birmingham, UK

Per Kristian Lehre  
School of Computer Science  
University of Birmingham, Birmingham, UK

## ABSTRACT

Competitive co-evolutionary algorithms (CoEAs) do not rely solely on an external function to assign fitness values to sampled solutions. Instead, they use the aggregation of outcomes from interactions between competing solutions allowing to rank solutions and make selection decisions. This makes CoEAs a useful tool for optimisation problems that have intrinsically interactive domains.

Over the past decades, many ways to aggregate the outcomes of interactions have been considered. At the moment, it is unclear which of these is the best choice. Previous research is fragmented and most of the fitness aggregation methods (fitness measures) proposed have only been studied empirically.

We argue that a proper understanding of the dynamics of CoEAs and their fitness measures can only be achieved through rigorous analysis of their behaviour. In this work we make a step towards this goal by using runtime analysis to study two commonly used fitness measures. We show a dichotomy in the behaviour of a  $(1, \lambda)$  CoEA when optimising a BILINEAR problem. The algorithm finds a Nash equilibrium efficiently if the worst interaction is used as a fitness measure but it takes exponential time w. o. p. if the average of all interactions is used instead.

## CCS CONCEPTS

• Theory of computation → Theory of randomized search heuristics.

## KEYWORDS

Runtime analysis, Competitive coevolution, MAXIMIN Optimisation

### ACM Reference Format:

Mario Alejandro Hevia Fajardo and Per Kristian Lehre. 2023. How Fitness Aggregation Methods Affect the Performance of Competitive CoEAs on Bilinear Problems. In *Genetic and Evolutionary Computation Conference (GECCO '23)*, July 15–19, 2023, Lisbon, Portugal. ACM, New York, NY, USA, 17 pages. <https://doi.org/10.1145/3583131.3590506>

## 1 INTRODUCTION

In biology the term co-evolution is used to describe a process where two or more species interact with each other reciprocally affecting their success and survival, and consequently their evolution. Co-evolution may happen in a mutualistic relationship, where both

species profit from the interaction or a competitive relationship where the species are adversaries (e. g. predator/prey, parasite/host).

Co-evolutionary algorithms (CoEAs) try to mimic biological co-evolution to solve optimisation problems [16]. As in their biological counterparts, they are divided in Cooperative CoEAs [13] and Competitive CoEAs depending on the kind of interactions the solutions take part in. We focus on Competitive CoEAs (we omit the *competitive* label and simply call them CoEAs).

CoEAs have found great success in several applications, ranging from test problems [1, 8] to real-world applications [4, 6, 18]. Despite their successes it has been well documented that CoEAs present pathological algorithm dynamics during the optimisation process [12, 16]. These pathological behaviours include, *evolutionary forgetting*: individuals with good characteristics are lost due to lack of selection pressure from the current opponent population, *cyclic dynamics*: traveling through some part of the search space more than once with apparent improvement, *disengagement*: one population overwhelmingly outperforms the other and the fitness gradient disappears, among others. Although these pathologies are well-known, due to the complex population dynamics of CoEAs, they have eluded our understanding. Albeit some remedies have been proposed [5, 6, 17, 18] we argue that without a rigorous understanding of what causes these harmful algorithm dynamics any attempt to solve them has a high likelihood to result futile.

CoEAs use populations of solutions that in each generation interact with each other and later assign a fitness to each solution based on the results (payoffs) of their interactions. A common practice is to aggregate the payoffs into a single numerical fitness value [16], e. g. by averaging the performance outcomes, use the maximum, minimum, median [20]<sup>1</sup> or more complicated statistics such as fitness sharing [17]. Unfortunately each fitness aggregation method may come with (sometimes harmful) biases depending on the optimisation problem being solved. For example, in cooperative CoEAs averaging interactions was observed to be harmful [21]. Bucci [2] even attributed some pathological behaviours encountered in CoEAs to the use of **any** aggregation method. At the moment, it is unclear when or if certain fitness aggregation methods (also called fitness measures) help the optimisation process or when they result in poor performance. Previous research is fragmented and all the fitness measures proposed have only been studied empirically<sup>2</sup>.

Due to their complexity, there is little rigorous understanding of the algorithm dynamics in (cooperative and competitive) CoEAs. Jansen and Wiegand [7] rigorously analysed the runtime of a cooperative CoEA on *separable* functions and showed that problem separability does not guarantee a speedup over traditional EAs. Lehre [9] analysed a population based CoEA on some instances of

Permission to make digital or hard copies of part or all of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for third-party components of this work must be honored. For all other uses, contact the owner/author(s).

GECCO '23, July 15–19, 2023, Lisbon, Portugal

© 2023 Copyright held by the owner/author(s).

ACM ISBN 979-8-4007-0119-1/23/07.

<https://doi.org/10.1145/3583131.3590506>

<sup>1</sup>studied in Cooperative CoEAs

<sup>2</sup>Other than small tailored examples showing fitness measures can be misleading [2]

BILINEAR, showing that given the correct parameters the algorithm finds an  $\varepsilon$ -approximation efficiently, but an incorrect parameter setting leads to exponential runtime. This work is a step forward towards a rigorous understanding of how different fitness measures can alleviate or aggravate the pathological behaviours of CoEAs.

We consider the  $(1, \lambda)$  CoEA (Algorithm 1) on a class of BILINEAR problems with an infinite discrete search space (c. f. Section 2.2). BILINEAR is a challenging class of MAXIMIN-problems because it has intransitive properties and the continuous versions are known to result in cyclic behaviour for some gradient-based algorithms [11, 14, 22]. Here, we ask whether the gradient-free  $(1, \lambda)$  CoEA is able to find a suitable solution despite the intransitivity of the problem and what is the role of the fitness measures in its performance.

In Sections 3 and 4 we characterise how the fitness measures assign fitness values based on the current population and how the mutation operator creates solutions as a stepping stone of the following theoretical analysis. Afterwards, in Section 5 we show that the  $(1, \lambda)$  CoEA using the average of interactions as fitness measure not only results in a cyclic behaviour, but every cycle the algorithm moves away from the optimum in expectation. This result in an exponential time (with respect to the initial distance to the optimum) to find an optimal solution w. o. p..

In sharp contrast, in Section 6 we show that the  $(1, \lambda)$  CoEA (creating offspring in a deterministic way) using the worst interaction as fitness measure is efficient, finding the optimum in polynomial time. We note that the algorithm still presents a cycling behaviour but the fitness measure helps alleviate the problem.

Finally, we present an experimental analysis where we test whether our theoretical analysis translates to other search domains. We observe that our theoretical analysis translates to other search domains. That is, for all search domains studied, using the average as a fitness measure results in exponential runtimes and using the minimum as a fitness measure results in efficient runtimes.

## 2 PRELIMINARIES

For any natural number  $n \in \mathbb{N}$  we define  $[n] := \{1, 2, \dots, n\}$ .

We study how different fitness aggregation methods (fitness measures) affect the performance of CoEAs on BILINEAR problems. In particular, we study the expected number of fitness evaluations (interactions) of the  $(1, \lambda)$  CoEA (Algorithm 1) with two different fitness measures on the class of MAXIMIN-optimisation problems called BILINEAR. Given a MAXIMIN-objective function  $g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$  (where  $\mathcal{X}, \mathcal{Y}$  are any *domain, strategy space or search space*), a MAXIMIN-optimisation problem is the problem of finding a solution  $x \in \mathcal{X}$  that maximises  $g$ , assuming that the adversary  $y \in \mathcal{Y}$  minimises  $g$  for that solution.

The  $(1, \lambda)$  CoEA as defined in Algorithm 1 optimises a MAXIMIN-optimisation problem. The  $(1, \lambda)$  CoEA uses any mutation operators  $\text{mut}_x\{\cdot\} : \mathcal{X} \rightarrow \mathcal{X}$  and  $\text{mut}_y\{\cdot\} : \mathcal{Y} \rightarrow \mathcal{Y}$ . In concordance to the algorithm, the class of BILINEAR problems can be defined in different search spaces too. We explore further this problem class in Section 2.2. For our theoretical analysis we consider  $\mathcal{X} = \mathcal{Y} = \mathbb{Z}$  and the mutation operator used is described in detail in Section 4.

We choose the search spaces  $\mathcal{X} = \mathcal{Y} = \mathbb{Z}$  for several reasons. The domain is discrete, therefore, gradient-based algorithms do not work denoting the importance of gradient-free algorithms such

as CoEAs. Despite being a discrete domain,  $\mathbb{Z}$  is infinite, which allows the creation of *real* unbiased mutation operators, in the sense that movement towards one direction of the search space is not limited by the search space itself making it less likely to happen. This reason is particularly important in this study, because we want to focus our analysis on how the fitness measures affect the performance of CoEAs and this allows us to decouple the behaviour of the selection mechanism (based on the fitness measures) from the inherent tendency away from the boundaries of the search space that a mutation operator in a finite search space have.

Due to space constraints, we removed some proofs from the paper; the detailed proofs can be found in the supplementary material.

### 2.1 The $(1, \lambda)$ CoEA

The  $(1, \lambda)$  CoEA uses a parent population of size one for each search space  $\mathcal{X}, \mathcal{Y}$ . Each generation it creates  $2\lambda$  offspring by mutating each parent  $\lambda$  times. Later it uses a fitness measure to assign a fitness value to every offspring in one population depending on their interactions with the other offspring population.

---

#### Algorithm 1: $(1, \lambda)$ CoEA

---

- 1 **Require:** MAXIMIN-objective function  $g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ .
  - 2 **Require:** Population size  $\lambda \in \mathbb{N}$ , fitness measures  $f_x : \mathcal{X} \times \mathcal{Y}^\lambda, f_y : \mathcal{Y} \times \mathcal{X}^\lambda$ , mutation operators  $\text{mut}_x\{\cdot\} : \mathcal{X} \rightarrow \mathcal{X}$ ,  $\text{mut}_y\{\cdot\} : \mathcal{Y} \rightarrow \mathcal{Y}$  and initialisation methods  $\text{init}_x\{\mathcal{X}\}, \text{init}_y\{\mathcal{Y}\}$
  - 3 **Initialization:**  $x_1 \leftarrow \text{init}_x\{\mathcal{X}\}$  and  $y_1 \leftarrow \text{init}_y\{\mathcal{Y}\}$ .
  - 4 **for**  $t \in \mathbb{N}$  **do**
  - 5     **Mutation:** **for**  $i \in \{1, \dots, \lambda\}$  **do**
  - 6          $P_t(i) \leftarrow \text{mut}\{x_t\}$  and  $Q_t(i) \leftarrow \text{mut}\{y_t\}$ ;
  - 7     **Selection:**
  - 8         Choose  $x_{t+1} \leftarrow \arg \max_{P_t(i)} \{f_x(P_t(i), Q_t)\}$ ;
  - 9         Choose  $y_{t+1} \leftarrow \arg \max_{Q_t(i)} \{f_y(Q_t(i), P_t)\}$ ;
- 

We consider two fitness measures  $f$ :

- Individual vs all (average)  $f^{\text{avg}}$ : Each individual in a population is evaluated against every other individual in its competing population, and the average of the evaluations is used to determine the individuals' fitness. That is,

$$f_X^{\text{avg}}(P_t(i), Q_t) = \frac{1}{\lambda} \sum_{j=1}^{\lambda} g(P_t(i), Q_t(j))$$

$$f_Y^{\text{avg}}(Q_t(i), P_t) = -\frac{1}{\lambda} \sum_{j=1}^{\lambda} g(P_t(j), Q_t(i)).$$

- Individual vs all (worst)  $f^{\text{wts}}$ : Each individual in a population is evaluated against every other individual in its competing population, and the worst of the evaluations is used to determine the individuals' fitness. That is,

$$f_X^{\text{wts}}(P_t(i), Q_t) = \min_{j \in [\lambda]} \{g(P_t(i), Q_t(j))\}$$

$$f_Y^{\text{wts}}(Q_t(i), P_t) = -\max_{j \in [\lambda]} \{g(P_t(j), Q_t(i))\}.$$

## 2.2 Bilinear

The class of BILINEAR problems is a simple and well-defined class of MAXIMIN-optimisation problems. The BILINEAR problems have been extensively used to understand the behaviour of MAXIMIN-optimisation algorithms (e.g. [11, 14, 22]) and it was recently used by Lehre [9] to analyse a population-based CoEA.

The general form of BILINEAR considers an  $n$ -dimensional (continuous or discrete) domain for the solutions  $x, y$ . Since we consider the 1-dimensional search spaces  $\mathcal{X} = \mathcal{Y} = \mathbb{Z}$ , we give a simpler form.

$$\text{BILINEAR}_{\alpha, \beta}(x, y) := xy - \alpha x - \beta y$$

The parameters  $\alpha$  and  $\beta$  denote where the MAXIMIN-solutions (also called Nash equilibria or optimal solutions) are found. We denote these solutions as OPT, that is,  $\text{OPT} := \{(x, y) \mid x = \beta \wedge y = \alpha\}$ .

During our analysis, we divide the search space into four *quadrants*. We say that a pair of search points  $(x, y)$  is in:

- the first quadrant if  $x < \beta \wedge y \geq \alpha$ ,
- the second quadrant if  $x \geq \beta \wedge y > \alpha$ ,
- the third quadrant if  $x > \beta \wedge y \leq \alpha$ , and
- the fourth quadrant if  $x < \beta \wedge y < \alpha$ .

## 2.3 Notation

By  $\mathcal{G}(p)$  we denote the geometric distribution with parameter  $p \in [0, 1]$ . The geometric distribution is often defined in two different ways, we define it as the probability distribution of the number of failures before the first success of Bernoulli trials with success probability  $p$ . Then, for  $X \sim \mathcal{G}(p)$  we have

$$\Pr[X = k] = (1 - p)^k p, \quad \mathbb{E}[X] = \frac{1 - p}{p}.$$

**Definition 2.1.** For populations  $P_t \in \mathcal{X}$  and  $Q_t \in \mathcal{Y}$  we define:

$$\begin{aligned} X_t &:= \sum_{x \in P_t} (x - \beta) & \bar{X}_t &:= \sum_{x \in P_t} |x - \beta| \\ Y_t &:= \sum_{y \in Q_t} (y - \alpha) & \bar{Y}_t &:= \sum_{y \in Q_t} |y - \alpha| \\ x_{\min} &:= \arg \min_{x \in P_t} \{x\} & x_{\max} &:= \arg \max_{x \in P_t} \{x\} \\ y_{\min} &:= \arg \min_{y \in Q_t} \{y\} & y_{\max} &:= \arg \max_{y \in Q_t} \{y\} \\ P_t^- &:= \{x \mid x \in P_t \wedge x \leq \beta\} & P_t^+ &:= \{x \mid x \in P_t \wedge x \geq \beta\} \\ Q_t^- &:= \{y \mid y \in Q_t \wedge y \leq \alpha\} & Q_t^+ &:= \{y \mid y \in Q_t \wedge y \geq \alpha\} \\ x_{\min}^+ &:= \arg \min_{x \in P_t^+} \{x\} & x_{\max}^- &:= \arg \max_{x \in P_t^-} \{x\} \\ y_{\min}^+ &:= \arg \min_{y \in Q_t^+} \{y\} & y_{\max}^- &:= \arg \max_{y \in Q_t^-} \{y\} \\ x_{\min}^* &:= \arg \min_{x \in P_t} \{|x - \beta|\} & y_{\min}^* &:= \arg \min_{y \in Q_t} \{|y - \alpha|\} \end{aligned}$$

## 3 ANALYSIS OF THE FITNESS MEASURES ON BILINEAR

In this section we focus our attention on how the fitness measures assign fitness to individual solutions. In particular, we explore what solutions have the *best* fitness with respect to each fitness measure on BILINEAR. We start with the fitness measure  $f^{\text{avg}}$ .

**Lemma 3.1** (Fitness Based on Averaging Interactions). *Consider two populations of solutions  $P_t$  and  $Q_t$  and the fitness measure  $f^{\text{avg}}$  on BILINEAR. Then  $f^{\text{avg}}$  assigns the highest fitness to all solutions in the sets  $\xi \subseteq P_t, \Upsilon \subseteq Q_t$  defined as,*

$$\xi = \begin{cases} \{x \mid x \in P_t \wedge x = x_{\min}\} & \text{if } Y_t < 0, \\ \{x \mid x \in P_t \wedge x = x_{\max}\} & \text{if } Y_t > 0, \\ P_t & \text{otherwise,} \end{cases}$$

and

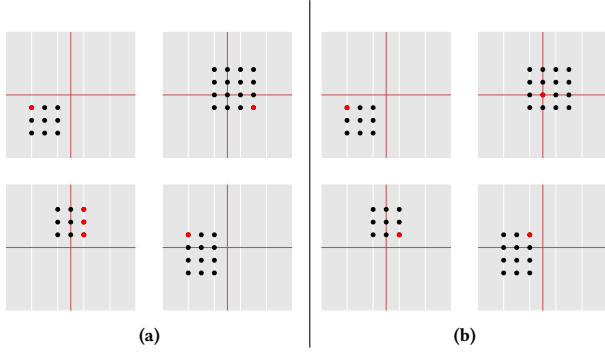
$$\Upsilon = \begin{cases} \{y \mid y \in Q_t \wedge y = y_{\max}\} & \text{if } X_t < 0, \\ \{y \mid y \in Q_t \wedge y = y_{\min}\} & \text{if } X_t > 0, \\ Q_t & \text{otherwise.} \end{cases}$$

The most important thing to note from Lemma 3.1 is that the decisions are made independent of the distance of the solutions to their respective optimum. Additionally, the highest fitness is assigned to solutions that at first glance seem to be detrimental to be selected. In fact, our analyses focus on the decisions made by the  $(1, \lambda)$  CoEA based on these *ill-informed* fitness values.

Lemma 3.2 describes the fitness measure  $f^{\text{wrs}}$ . The behaviour of this fitness measure is noticeably more complex than  $f^{\text{avg}}$ . Due to the number of different cases we only display the cases for  $x$  in Lemma 3.2. The cases for  $y$  are the same, exchanging  $x, P_t$  and  $\alpha$  for  $y, Q_t$  and  $\beta$  and vice versa. The full lemma and its proof can be found in the supplementary material.

**Lemma 3.2** (Fitness Based on Worst Interactions). *Consider two populations of solutions  $P_t$  and  $Q_t$  and the fitness measure  $f^{\text{wrs}}$  on BILINEAR. Then  $f^{\text{wrs}}$  assigns the highest fitness to all solutions in the set  $\xi \subseteq P_t$  defined as,*

$$\xi = \begin{cases} \{x \mid x \in P_t \wedge x = x_{\max}^-\} & \text{if } |Y_t| \neq \bar{Y}_t \wedge |X_t| \neq \bar{X}_t \wedge \\ & (\beta - x_{\max}^-)(\alpha - y_{\max}) \geq \\ & (\beta - x_{\min}^+)(\alpha - y_{\min}), \quad (1) \\ \{x \mid x \in P_t \wedge x = x_{\min}^+\} & \text{if } |Y_t| \neq \bar{Y}_t \wedge |X_t| \neq \bar{X}_t \wedge \\ & (\beta - x_{\max}^-)(\alpha - y_{\max}) \leq \\ & (\beta - x_{\min}^+)(\alpha - y_{\min}), \quad (2) \\ \{x \mid x \in P_t \wedge x = x_{\min}^*\} & \text{if } |Y_t| \neq \bar{Y}_t \wedge |X_t| = \bar{X}_t, \quad (3) \\ P_t^+ & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t > 0 \wedge \\ & \exists y \in Q_t, y = \alpha \wedge P_t^+ \neq \emptyset, \quad (4) \\ \{x \mid x \in P_t \wedge x = x_{\min}^*\} & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t > 0 \wedge \\ & \exists y \in Q_t, y = \alpha \wedge P_t^+ = \emptyset, \quad (5) \\ P_t^- & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t < 0 \wedge \\ & \exists y \in Q_t, y = \alpha \wedge P_t^- \neq \emptyset, \quad (6) \\ \{x \mid x \in P_t \wedge x = x_{\min}^*\} & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t < 0 \wedge \\ & \exists y \in Q_t, y = \alpha \wedge P_t^- = \emptyset, \quad (7) \\ \{x \mid x \in P_t \wedge x = x_{\max}\} & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t > 0 \wedge \\ & \forall y \in Q_t, y \neq \alpha, \quad (8) \\ \{x \mid x \in P_t \wedge x = x_{\min}\} & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t < 0 \wedge \\ & \forall y \in Q_t, y \neq \alpha, \quad (9) \\ P_t & \text{if } y = \alpha \forall y \in Q_t, \quad (10) \end{cases}$$



**Figure 1: Example populations  $P \times Q$  with the pair(s) of individuals with highest fitness value highlighted (red point) for fitness measures  $f^{\text{avg}}$  (a) and  $f^{\text{wts}}$  (b).  $\alpha$   $\beta$  are denoted by the red lines and the optimum is at the crossing of these lines.**

Figure 1 shows some example populations and their highest individuals for fitness measures  $f^{\text{avg}}$  (a) and  $f^{\text{wts}}$  (b). The most interesting example is when  $P \times Q$  comprises the four quadrants and the optimal solution is included (top right in 1 (a) and 1 (b)). In this case  $f^{\text{wts}}$  correctly assigns the highest fitness to the optimum but  $f^{\text{avg}}$  does not. When  $P \times Q$  comprises two quadrants (bottom left and right in 1 (a) and 1 (b))  $f^{\text{avg}}$  tend to assign the highest fitness to individuals farther away from the optimum than the ones chosen by  $f^{\text{wts}}$ . Finally if  $P \times Q$  comprises one quadrant (top left in 1 (a) and 1 (b)) both fitness measures behave identically: the individual with highest fitness is always the corner *ahead* in a clockwise direction.

#### 4 ANALYSIS OF THE MUTATION OPERATOR

The  $(1, \lambda)$  CoEA analysed here uses the following simple mutation operator, which is called geometric mutation. For a parent  $x$  it creates the  $i$ -th offspring  $x'$  by sampling  $X_i \sim \mathcal{G}(p)$  and adding or subtracting that value to the parent. To simplify the analysis the mutation operator creates exactly  $\lambda/2$  offspring as  $x' = x + X_i$  and the other  $\lambda/2$  offspring as  $x' = x - X_i$ . This could be randomised, maintaining the same expected number of offspring increasing and decreasing value from the parent as follows: with probability  $1/2$  the operator adds  $X_i$  and subtracts  $X_i$  otherwise. Although we believe our analysis holds for the randomised case, we decided not to add it to avoid over-complicating the already complex analysis.

In Section 3 we have seen that the fitness measures assign fitness to an individual based on the sum of distances to  $\alpha$  or  $\beta$ . To this effect the following lemma is useful to understand how much this sum can change.

**Lemma 4.1.** *Let  $h \in \mathbb{N}$  and  $Y_i$  be i.i.d. random variables sampled from  $\mathcal{G}(p)$ . Then,*

$$\Pr \left[ \sum_{i=1}^{\psi} (Y_i + h) - \sum_{i=\psi+1}^{2\psi} (Y_i - h) \leq 0 \right] = e^{-\Omega(h\psi)}.$$

Another important thing that we learnt in Section 3 is that the offspring with highest fitness is often the farthest away from the parent (with respect to one of the directions). Therefore, we need to characterise the behaviour of the maximum value of several

geometric random variables. In the following lemma we study its expected value.

**Lemma 4.2.** *Let  $0 < p < 1$ ,  $\psi \in \mathbb{N}$ ,  $X_i \sim \mathcal{G}(p)$  and  $X^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$  then*

$$\left(1 - e^{-\frac{1}{1-p}}\right) \log_{\frac{1}{1-p}} \psi \leq \mathbb{E} \left[ X^{(\psi)} \right] \leq \log_{\frac{1}{1-p}} \psi + \frac{1-p}{p} \quad (11)$$

The expected value is not sufficient to fully understand the behaviour of the algorithm. In the following we show tail bounds on the maximum value of several geometric random variables.

**Lemma 4.3.** *Let  $0 < p < 1$  and  $c > 0$ . Let  $\delta > 0$ ,  $\psi \in \mathbb{N}$ ,  $X_i \sim \mathcal{G}(p)$  and  $X^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$  then*

$$\Pr \left[ X^{(\psi)} \geq (1 + \delta) \frac{\ln \psi}{p} \right] \leq \min \{ \psi^{-\delta}, 1 \} \quad (12)$$

$$\Pr \left[ X^{(\psi)} \geq \log_{\frac{1}{1-p}} c \psi \right] \leq 1/c \quad (13)$$

$$\Pr \left[ X^{(\psi)} < (1 - \delta) \frac{p \ln \psi}{1-p} \right] \leq \exp \left( -\psi^\delta \right) \quad (14)$$

Algorithm 1 uses two competing populations and while one population moves towards the optimum the other can move away from it. In these cases we need to understand what is the expected movement overall and what is the probability of this deviating from its expectation. The following lemmas analyse these cases.

**Lemma 4.4.** *Let  $0 < p < 1$ . Let  $\psi \in \mathbb{N}$ ,  $\tau \in \mathbb{N}$ ,  $X_i, Y_i$  be independent and identically distributed (i.i.d.) random variables sampled from  $\mathcal{G}(p)$ ,  $X_j^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$  and  $Y_j^{(\psi)} := \max_{i \in [\psi]} \{Y_i\}$ . Let*

$$Z := \sum_{j=1}^{\tau} \left( X_j^{(\psi)} - Y_j^{(\psi)} \right). \text{ Then } \mathbb{E}[Z] = 0.$$

**Lemma 4.5.** *Let  $0 < \varepsilon < p < 1$ . Let  $|\eta| < p - \varepsilon$ ,  $\psi \in \mathbb{N}$ ,  $\tau \in \mathbb{N}$ ,  $X_i \sim \mathcal{G}(p)$  and  $X^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$ . Then  $\mathbb{E} \left[ e^{\eta X^{(\psi)}} \right] \leq \frac{\psi}{\varepsilon}$  and  $\mathbb{E} \left[ e^{-\eta X^{(\psi)}} \right] \leq \psi \left( 1 + \frac{1}{\varepsilon} \right)$ .*

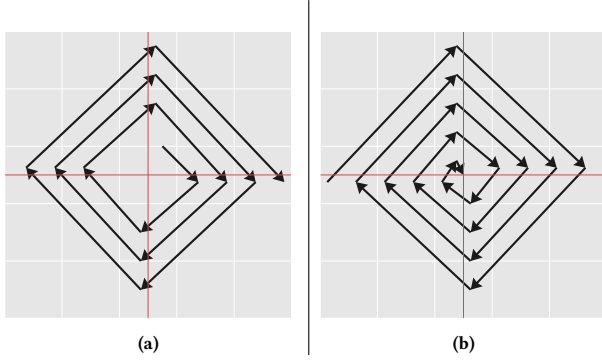
**Lemma 4.6.** *Let  $0 < p < 1$ . Let  $\psi^6 \geq \left( 1 + \frac{2}{p} \right)^2$  with  $\psi \in \mathbb{N}$ . Let  $r' := \frac{8r\sqrt{2 \ln \psi}}{p}$ ,  $r \geq 2$ ,  $j \in \mathbb{N}_0$ ,  $\tau \in \mathbb{N}$ ,  $X_i, Y_i \sim \mathcal{G}(p)$  be i.i.d. random variables,  $X^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$  and  $Y^{(\psi)} := \max_{i \in [\psi]} \{Y_i\}$ . Let  $Z := \sum_{i=1}^{\tau} (X^{(\psi)} - Y^{(\psi)})$ . Then,*

$$\Pr \left[ |Z| \geq jr' \right] \leq e^{-jr'}.$$

#### 5 AVERAGING INTERACTIONS IS INEFFICIENT

In this section we study the fitness measure  $f^{\text{avg}}$ . We show that the  $(1, \lambda)$  CoEA using  $f^{\text{avg}}$  takes exponential time to reach the optimum.

**THEOREM 5.1.** *Consider Algorithm 1 with fitness measure  $f^{\text{avg}}$  and geometric mutation operator on BILINEAR. Define  $T = \min \{ t \mid \text{OPT} \cap (P_t \times Q_t) \neq \emptyset \}$ . Let  $d := |x_1 - \beta| + |y_1 - \alpha|$ . If the following conditions hold:*



**Figure 2: Expected behaviour of  $(1, \lambda)$  CoEA with fitness measure  $f^{\text{avg}}$  (a) and fitness measure  $f^{\text{wrs}}$  (b) on BILINEAR.**

- $\lambda \leq d^{a-2}$  for some constant  $a > 2$ .
- $\left(1 - e^{-\frac{1}{1-p}}\right) \log_{\frac{1}{1-p}} \lambda/2 \geq 2$

Then,  $T \geq e^{\Omega(d)}$  with probability  $1 - e^{-\Omega(d)}$ .

The main idea of the proof of Theorem 5.1 is that the algorithm moves through each quadrant in a clockwise manner. While inside a quadrant the algorithm maintains its distance to the optimum in expectation and each time it moves from one quadrant to the other the last generation continues in the wrong direction moving away from the optimum. Figure 2 (a) visualises this behaviour.

To facilitate the analysis we divide a run of the algorithm into different intervals that denote when the algorithm moves from one quadrant to the other (not necessarily in a clockwise manner).

**Definition 5.2 (Blocks).** Let  $\tau_i = \min\{t > \tau_{i-1} \mid (x_t - \beta)(x_{\tau_{i-1}} - \beta) < 0 \vee (y_t - \alpha)(y_{\tau_{i-1}} - \alpha) < 0\}$  and  $\tau_0 := 0$ . We divide a run into generation intervals  $(\tau_k, \tau_{k+1}]$  that we call blocks.

The first step is to show that the algorithm does not spend too much time in one block. This is shown in the following lemma.

**Lemma 5.3.** Let  $c > 0$ ,  $a > 2$  be constants such that for some  $d > 0$ ,  $d^a - cd = \Omega(d^{a-1})$ . Consider the  $(1, \lambda)$  CoEA with  $\lambda \leq d^{a-2}$  starting a block with a Manhattan distance  $cd > 0$ . The probability that the algorithm spends more than  $d^a$  generations during a block is  $e^{-\Omega(d)}$ .

**PROOF.** By the definition of a block once the algorithm moves to another quadrant a block finishes. Here we omit all possibilities of moving to another quadrant except by moving to the quadrant that is clockwise to the current one. Since the initial Manhattan distance is  $cd$  the algorithm needs to move in one direction (clockwise) by at least  $cd$ .

To explain the computations we assume that we are in the second quadrant ( $x_t - \beta > 0$  and  $y_t - \alpha > 0$ ) and the algorithm moves downwards to the next quadrant in expectation, that is  $y$  is decreasing until it reaches  $y < \alpha$ . Thanks to the symmetry of the behaviour of the algorithm in the quadrants, the computations hold for all quadrants.

To fit the perspective of the additive drift tail bounds [10, Theorem 2.4.7] we use a potential function  $h(y_t) := \min\{0, y_t - \alpha\}$ . Let  $\Delta := h(y_t) - h(y_{t+1})$  and note that  $\Delta \geq 0$  as long as  $X_t > 0$  because the algorithm creates at least  $\lambda/2$  offspring with the same value

of  $y_t$  or less and by Lemma 3.1 the algorithm selects the offspring with the smallest value. If  $X_t \leq 0$ , the opposite is true, and then  $\Delta \leq 0$ .

Recall that  $X_t := \sum_{x \in P_t} (x - \beta)$ . Let  $h := x_t - \beta > 0$ . Then  $X_t$  is equivalent to

$$\sum_{i=1}^{\lambda/2} (X_i + h) - \sum_{i=1}^{\lambda/2} (X_i - h),$$

where  $X_i$  are i.i.d. random variables sampled from  $\mathcal{G}(p)$ . Then, by Lemma 4.1,  $\Pr[X_t \leq 0 \mid x_t - \beta > 0] = e^{-\Omega(\lambda)}$ .

By Lemma 4.2 using  $\psi = \lambda/2$  we obtain

$$\begin{aligned} E[\Delta] &\geq \Pr[X_t \leq 0 \mid x_t - \beta > 0] E[\Delta \mid X_t \leq 0] \\ &\quad + \Pr[X_t > 0 \mid x_t - \beta > 0] E[\Delta \mid X_t > 0] \\ &\leq e^{-\Omega(\lambda)} \left( \log_{\frac{1}{1-p}}(\lambda/2) + \frac{1-p}{p} \right) \\ &\quad + \left(1 - e^{-\Omega(\lambda)}\right) \left(1 - e^{-\frac{1}{1-p}}\right) \log_{\frac{1}{1-p}} \lambda/2 \\ &= \Omega(\log \lambda) \end{aligned}$$

This shows condition (2.24) from [10, Theorem 2.4.7].

Independent on if the algorithm moves upward or downward the  $\Pr[|\Delta| > j] \leq \Pr[Z \geq j]$  with  $Y_i \sim \mathcal{G}(p)$  and  $Z := \max_{i \in [\lambda/2]} \{Y_i\}$ .

Therefore, by Lemma 4.3 (13) with  $c = 2e^j/\lambda$  we obtain,

$$\Pr[|\Delta| > j] \leq \frac{\lambda}{2e^j},$$

and the last condition (2.21) from [10, Theorem 2.4.7] is met with  $\eta = 1 - e$  and  $r = \lambda/2$ .

Let,  $\tau$  be the time the algorithm spends in a block, then by the additive drift tail bounds.

$$\begin{aligned} \Pr[\tau \geq d^a] &\leq \exp\left(\left(-\frac{(1-e)(d^a - cd)}{8}\right)\left(\frac{(1-e)^2(d^a - cd)}{16\lambda cd}\right)\right) \\ &= \exp\left(-\Omega(d^{a-1})\right) = e^{-\Omega(d)}. \quad \square \end{aligned}$$

Now, we show that the algorithm does not *jump* far away from its current parents during one generation.

**Lemma 5.4.** Let  $d > 0$ . The probability that a generation from Algorithm 1 using the geometric mutation operator starts with parents  $x, y$  and creates a pair of points  $x', y'$  with a Manhattan distance from their parents  $|x - x'| + |y - y'| \geq d$  is at most  $\lambda^3 e^{-dp}$ . If  $\lambda^3 = e^{o(d)}$  this is  $e^{-\Omega(d)}$ .

**PROOF.** By Lemma 4.3 with  $\psi = \lambda$ , the probability of creating offspring  $x'$  and  $y'$  with distance  $|x - x'|$  and  $|y - y'|$  from  $x$  and  $y$  respectively is

$$\min\left\{\lambda^{1-|x-x'|} \frac{p}{\ln \lambda}, 1\right\} \cdot \min\left\{\lambda^{1-|y-y'|} \frac{p}{\ln \lambda}, 1\right\} \quad (15)$$

If none of the two expressions are minimised by 1, then this becomes

$$\lambda^{2-(|x-x'|+|y-y'|)} \frac{p}{\ln \lambda} < \lambda^{2-cd} \frac{p}{\ln \lambda} = \lambda^2 e^{-dp}.$$

On the other hand if one of the expressions is minimised by 1, then Equation (15) implies that either  $|x - x'| \leq \ln \lambda/p$  and  $|y - y'| \geq$

$d - \ln \lambda/p$  or  $|y - y'| \leq \ln \lambda/p$  and  $|x - x'| \geq d - \ln \lambda/p$ . Therefore, Equation (15) yields

$$\lambda^{2-(d-\frac{\ln \lambda}{p})\frac{p}{\ln \lambda}} = \lambda^{3-cd\frac{p}{\ln \lambda}} = \lambda^3 e^{-dp}.$$

Both cases are at most  $\lambda^3 e^{-dp}$ . If  $\lambda^3 = e^{o(d)}$  then there is a constant  $c > 0$  for which  $\lambda^3 e^{-dp} \leq e^{-dc}$ .  $\square$

The main part of the analysis is to show that there is a large region (of size  $\Theta(d)$ ) in the search space where the algorithm moves away from the optimum in expectation after each block.

**Lemma 5.5.** *Let  $c > 0$  be a constant. Let  $(1 - e^{-\frac{1}{1-p}}) \log_{\frac{1}{1-p}} \lambda/2 \geq 2$ . Let  $M_k := |x_{\tau_k} - \beta| + |y_{\tau_k} - \alpha| > cd$  be the Manhattan distance at the beginning of block  $k$ . Then, there exists a constant  $\delta > 0$  such that  $E[M_{k+1} - M_k \mid M_k > cd] \geq \delta$ .*

**PROOF.** By Lemma 5.3 there are at most  $d^a$  generations (for some constant  $a > 2$ ) in one block with probability  $1 - e^{-\Omega(d)}$ . We call  $E_1$  to the event that a block takes longer than  $d^a$  generations. If event  $E_1$  happens we pessimistically assume that the optimum is found during the block, that is  $M_{k+1} = 0$ .

By the dynamics of the algorithm on the BILINEAR problem every block is spent in a quadrant of the search space located clockwise from the previous block, unless there was a generation within the block with at least one pair of search points from  $P_t \times Q_t$  in all of the four quadrants of the search space. We call  $E_2$  the event that during the current block there is a generation  $t$  where the algorithm creates at least one pair of search points from  $P_t \times Q_t$  in all of the four quadrants and pessimistically assume that if this happens the optimum is found during the block. Therefore  $\overline{E_2}$  guarantees that the next block is located clockwise from the previous block.

If event  $E_1$  does not happen, by Lemma 5.4 the probability that event  $E_2$  happens during one generation is  $e^{-\Omega(d)}$ . By a union bound over  $d^a$  generations, the probability of  $E_2$  is still  $e^{-\Omega(d)}$ . Then

$$\Pr[E_1 \vee E_2] E[M_{k+1} - M_k \mid M_k > cd \wedge (E_1 \vee E_2)] \geq -cde^{-\Omega(d)} \quad (16)$$

From now on we consider only cases where events  $E_1$  and  $E_2$  does not happen. For simplicity, in the following we also assume that the block  $k$  is in the second quadrant of the search space and note that thanks to the symmetry of the behaviour of the algorithm the computation holds for all quadrants.

$$\text{Recall the definitions } X_t := \sum_{x \in P_t} (x - \beta) \text{ and } Y_t := \sum_{y \in Q_t} (y - \alpha).$$

We assume that if  $x_t - \beta > 0$  then  $X_t > 0$  because of the following: if this is not the case during one generation, (with block  $k$  in the second quadrant) then by Lemma 3.1 the algorithm would choose either the pair of offspring farthest away from the optimum (if  $X_t < 0$ ) or a pair of offspring that is at least as far as the one that would be chosen otherwise (if  $X_t = 0$ ). Given this assumption, by Lemma 4.4 all generations but the last one have an expected change in distance to the optimum of 0.

There are two different cases for the expected change in distance depending on where the last generation  $t = \tau_{k+1} - 1$  of a block starts. Specifically, the two cases are  $y_t - \alpha = 0$  and  $y_t - \alpha > 0$ . We denote these cases as  $E_3$  and  $E_4$ .

If the last generation starts with  $y_t - \alpha = 0$  (event  $E_3$ ) then there is a probability  $\xi$  that  $Y_t = 0$  and  $x_{t+1}$  is chosen uniformly at random from  $P_t$ . Since the offspring in  $P_t$  have the same probability of having a value  $x_t + i$  and  $x_t - i$  for all  $i \in \mathbb{N}$  then in expectation  $x_t = x_{t+1}$  and the pair  $x_{t+1}, y_{t+1}$  are farther from the optimum by  $y_t - y_{t+1}$  which is at least  $(1 - e^{-\frac{1}{1-p}}) \log_{\frac{1}{1-p}} \lambda/2$  in expectation (Lemma 4.2). There is also a probability  $(1 - \xi)/2$  of  $Y_t > 0$  and  $(1 - \xi)/2$  of  $Y_t < 0$ . In the former case by Lemma 3.1 the algorithm selects  $x_{t+1} = \arg \max_{x \in P_t} \{x\}$ , that is, the offspring farthest away from the optimum. Pessimistically assuming that  $y_t = y_{t+1}$ , the pair  $x_{t+1}, y_{t+1}$  are farther from the optimum by at least  $(1 - e^{-\frac{1}{1-p}}) \log_{\frac{1}{1-p}} \lambda/2$  (Lemma 4.2). Finally, if  $Y_t < 0$  in expectation the distance to the optimum is at least the same, therefore,

$$\begin{aligned} E[M_{k+1} - M_k \mid M_k > cd \wedge E_3] & \geq \left( \frac{1-\xi}{2} + \xi \right) (1 - e^{-\frac{1}{1-p}}) \log_{\frac{1}{1-p}} \lambda/2 \\ & \geq \frac{1}{2} (1 - e^{-\frac{1}{1-p}}) \log_{\frac{1}{1-p}} \lambda/2 \geq 1 \end{aligned}$$

For event  $E_4$ , we denote  $h := y_t - \alpha > 0$ . As event  $E_3$  there are three outcomes depending on the value of  $Y_t$ . Therefore, we use Lemma 4.1 to compute  $\Pr[Y_t \leq 0 \mid y_t - \alpha > 0]$ . Recall that  $Y_t := \sum_{y \in Q_t} (y - \alpha)$ . Then  $Y_t$  is equivalent to

$$\sum_{i=1}^{\lambda/2} (Y_i + h) - \sum_{i=\lambda/2+1}^{\lambda} (Y_i - h),$$

where  $Y_i$  are i.i.d. random variables sampled from  $\mathcal{G}(p)$ . Then, by Lemma 4.1

$$\Pr[Y_t \leq 0 \mid y_t - \alpha > 0] = e^{-\Omega(\lambda)}.$$

If  $Y_t < 0$ , by Lemmas 3.1 and 4.2  $x_{t+1}$  moves towards the optimum by at most  $(\log_{\frac{1}{1-p}} \lambda/2 - \frac{1-p}{p})$  in expectation. If  $Y_t = 0$ ,  $x_t = x_{t+1}$  in expectation. For the  $y$  direction, we are interested in the change in distance away from the optimum solution given that  $y_t > \alpha$  and  $y_{t+1} < \alpha$ , that is,  $(\alpha - y_{t+1}) - (y_t - \alpha)$ . The event  $Y_t \leq 0$  is positively correlated with  $\alpha - y_{t+1}$  because it implies that there are more offspring with values less than  $\alpha$  than the expectation and they are farther from the optimum. For the same reason it is also negatively correlated with  $y_t - \alpha$ . Therefore we can bound the change in distance to the optimum in the  $y$  direction by the unconditional expected step size given by Lemma 4.2. Adding both directions results in a decrease in distance to the optimum by at most  $2 \left( \log_{\frac{1}{1-p}} \lambda/2 - \frac{1-p}{p} \right)$  in expectation for both  $Y_t < 0$  and  $Y_t = 0$ .

Finally, for  $Y_t > 0$  with  $y_t - \alpha > 0$  we denote  $\Delta_y := y_{t+1} - y_t$  and  $\Delta_x := x_{t+1} - x_t$ . We aim to show that  $E[\Delta_y] = E[\Delta_x] - o(1)$  which in conjunction with the fact that  $y_{t+1} > \alpha$  imply that in the worst case (best case for the algorithm) the Manhattan distance to the optimum increases by at least  $2 - o(1)$ .

Given the condition  $Y_t > 0$  we have  $E[\Delta_y] = E[Y^{(\lambda/2)} \mid Y_t > 0]$ , where  $Y^{(\lambda/2)}$  is the sum of  $\lambda/2$  geometrically distributed random

variables. By the law of total expectation,

$$\mathbb{E}\left[Y^{(\lambda/2)} \mid Y_t > 0\right] = \frac{\mathbb{E}\left[Y^{(\lambda/2)}\right] - \mathbb{E}\left[Y^{(\lambda/2)} \mid Y_t \leq 0\right] \cdot \Pr[Y_t \leq 0]}{\Pr[Y_t > 0]}$$

We note that the condition  $Y_t \leq 0$  is equivalent to

$$\sum_{i=1}^{\lambda/2} (Y_i + h) - \sum_{i=\lambda/2+1}^{\lambda} (Y_i - h) \leq 0.$$

If all  $Y_i$  with  $i \in (\lambda/2, \lambda]$  are greater or equal to  $2h + \max_{i \in [1, \lambda/2]} \{Y_i\}$  then  $Y_t \leq 0$  is guaranteed. Thanks to the *forgetfulness* of the geometric distribution, for  $Y_i \sim \mathcal{G}(p)$ ,  $\mathbb{E}[Y_i \mid Y_i \geq j] = j + \mathbb{E}[Y_i]$ . Therefore,

$$\mathbb{E}\left[Y^{(\lambda/2)} \mid Y_t \leq 0\right] \leq 2h + 2\mathbb{E}\left[Y^{(\lambda/2)}\right] \leq 4h\mathbb{E}\left[Y^{(\lambda/2)}\right],$$

where the last inequality holds because  $\mathbb{E}\left[Y^{(\lambda/2)}\right] > 1$  by the assumptions and Lemma 4.2 and  $h \geq 1$ . As mentioned before, by Lemma 4.1  $\Pr[Y_t \leq 0] = e^{-\Omega(h\lambda)}$  and

$$\mathbb{E}\left[Y^{(\lambda/2)} \mid Y_t \leq 0\right] \cdot \Pr[Y_t \leq 0] \leq 4h\mathbb{E}\left[Y^{(\lambda/2)}\right] e^{-\Omega(h\lambda)} = o(1),$$

Hence,

$$\mathbb{E}\left[Y^{(\lambda/2)} \mid Y_t > 0\right] \geq \frac{\mathbb{E}\left[Y^{(\lambda/2)}\right]}{1 - e^{-\Omega(h\lambda)}} - o(1) = \mathbb{E}\left[Y^{(\lambda/2)}\right] - o(1),$$

and the Manhattan distance to the optimum increases by at least  $2 - o(1)$ . Joining the three cases for event  $E_4$ , we obtain

$$\begin{aligned} & \mathbb{E}[M_{k+1} - M_k \mid M_k > cd \wedge E_4] \\ & \geq -2 \left( \log \frac{1}{1-p} \lambda/2 - \frac{1-p}{p} \right) e^{-\Omega(\lambda)} + (2 - o(1))(1 - e^{-\Omega(\lambda)}) \geq \delta'. \end{aligned}$$

Where  $0 < \delta' < 2 - o(1)$  is a constant independent of  $d$  and  $\lambda$ .

Joining all events together yields,

$$\begin{aligned} & \mathbb{E}[M_{k+1} - M_k \mid M_k > cd] = \\ & \Pr[E_1 \vee E_2] \mathbb{E}[M_{k+1} - M_k \mid M_k > cd \wedge (E_1 \vee E_2)] + \\ & \Pr[E_3] \mathbb{E}[M_{k+1} - M_k \mid M_k > cd \wedge E_3] + \\ & \Pr[E_4] \mathbb{E}[M_{k+1} - M_k \mid M_k > cd \wedge E_4] \\ & \geq -cde^{-\Omega(d)} + \Pr[E_3] + \delta' \Pr[E_4] \\ & \geq -cde^{-\Omega(d)} + \delta' \Pr[E_3 \vee E_4] / 2 \\ & \geq -cde^{-\Omega(d)} + \delta' (1 - e^{-\Omega(d)}) / 2 \geq \delta, \end{aligned}$$

where  $\delta > 0$  is a constant independent of  $d$  and  $\lambda$ .  $\square$

With the previous helper lemmas we are now in position to prove the main theorem of this section.

**PROOF OF THEOREM 5.1.** We aim to use the negative drift theorem with scaling [15, Theorem 2]. By Lemma 5.5 every block starting with a Manhattan distance  $cd$  from the optimum has at least constant negative drift  $\delta > 0$ . This meets condition (1) of [15, Theorem 2].

For Condition (2), we use Lemma 4.6 with  $r' = \frac{16\sqrt{2\ln\lambda/2}}{p}$ . Then, for all blocks excluding the last generation (from  $t = \tau_k + 1$  to  $t = \tau_{k+1} - 1$ ), the following holds,

$$\Pr\left[|M_{\tau_{k+1}} - M_{\tau_{k+1}-1}| \geq jr'\right] \leq e^{-2j}$$

Then, we need to only consider the last generation. In the worst case both  $x_{\tau_{k+1}}$  and  $y_{\tau_{k+1}}$  move towards/away from the optimum. By Lemma 5.4 the probability of creating an offspring at a distance at least  $\ell$  is at most  $\lambda^3 e^{-\ell p} = e^{3\ln\lambda - \ell p}$ . Using  $\ell = \frac{4j\ln\lambda}{p}$  yields,

$$e^{3\ln\lambda - \ell p} = e^{3\ln\lambda - 4j\ln\lambda} \leq e^{-j\ln\lambda}.$$

Then choosing  $r = r' + \frac{4\ln\lambda}{p}$ ,

$$\Pr\left[|M_{\tau_{k+1}} - M_{\tau_{k+1}-1}| \geq jr\right] \leq e^{-2j} + e^{-j\ln\lambda} \leq e^{-j}.$$

We note that  $r^2 = O(\log^2 \lambda)$ . Then Condition (3) of [15, Theorem 2] holds thanks to the assumption  $\lambda = e^{o(d)}$ .  $\square$

## 6 FITNESS BASED ON WORST INTERACTIONS IS EFFICIENT

Owing to the large number of different behaviours the fitness measure  $f^{\text{WTS}}$  can take, its analysis is much more involved. In Section 4 we have seen that the geometric mutation operator generates offspring around the parents in a *square* pattern (as the example populations shown in Figure 1 (b)), and deviations from the expected behaviour have exponential decay. Although we believe the main result of this section (Theorem 6.1) holds for the geometric mutation operator, we simplify the computations by assuming that the offspring in each generation are always created around the parents in an exact *square* pattern. For the readers' convenience we present the pseudo-code of this instantiation of the  $(1, \lambda)$  CoEA in Algorithm 2.

---

### Algorithm 2: Deterministic $(1, \lambda)$ CoEA

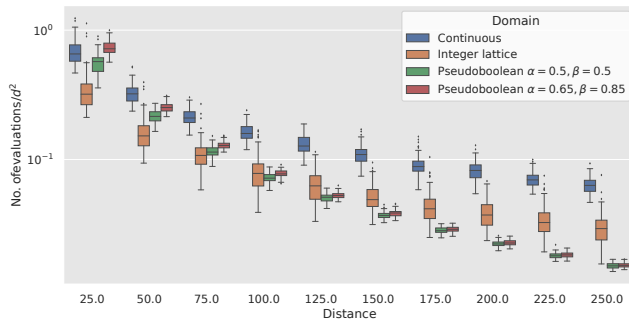
---

- 1 **Require:** Max-min-objective function  $g : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ .
  - 2 **Require:** Population size  $\lambda = 2r + 1$  with radius  $r \in \mathbb{N}$ , and fitness measures  $f_x : \mathcal{X} \times \mathcal{Y}^\lambda, f_y : \mathcal{Y} \times \mathcal{X}^\lambda$
  - 3 **Initialization:** Sample  $x_1 = 0$  and  $y_1 = 0$ .
  - 4 **for**  $t \in \mathbb{N}$  **do**
  - 5     **Mutation:** **for**  $i \in \{0, \dots, \lambda - 1\}$  **do**
  - 6          $P_t(i) \leftarrow x_t - r + i;$
  - 7          $Q_t(i) \leftarrow y_t - r + i;$
  - 8     **Selection:**
  - 9     Choose  $x_{t+1} \leftarrow \arg \max_{P_t(i)} \{f_x(P_t(i), Q_t)\};$
  - 10     Choose  $y_{t+1} \leftarrow \arg \min_{Q_t(i)} \{f_y(Q_t(i), P_t)\};$
- 

**THEOREM 6.1.** Consider Algorithm 2 with fitness measure  $f^{\text{WTS}}$  and radius  $r \geq 1$  on *BILINEAR*. Define  $T = \min\{t \mid \text{OPT} \cap (P_t \times Q_t) \neq \emptyset\}$ . Let  $M_1 := |x_1 - \beta| + |y_1 - \alpha| > r$  be the initial Manhattan distance to the optimum. Then,  $\mathbb{E}[T] \leq \frac{M_1^2}{r} - r$ . If  $M_t \leq r$  then  $T = 1$ .

## 7 EXPERIMENTS

Although our theoretical results have given us important insights of how the fitness measures affect the performance of CoEAs, they are limited to only one search space ( $\mathcal{X} = \mathcal{Y} = \mathbb{Z}$ ). It is unclear if our results translate to other search spaces. In this section, we conduct an experimental analysis that aims to complement our



**Figure 3: Runtime of the  $(1, \lambda)$  CoEA (Algorithm 1) with fitness measure  $f^{\text{wrs}}$  on BILINEAR.**

theoretical results with precise runtime results for the  $(1, \lambda)$  CoEA (Algorithm 1) with different fitness measures on BILINEAR problems defined over three different search spaces ( $\mathcal{X} = \mathcal{Y} = \mathbb{Z}$ ,  $\mathcal{X} = \mathcal{Y} = \mathbb{R}$  and  $\mathcal{X} = \mathcal{Y} = \{0, 1\}^n$ ). For the search space  $\mathcal{X} = \mathcal{Y} = \{0, 1\}^n$  we use the definition of BILINEAR used by [9] with  $n = 500$  in all experiments and two parameter values ( $\alpha = \beta = 0.5$  and  $\alpha = 0.8, \beta = 0.75$ ). We note that the mutation operator in the case  $\alpha = \beta = 0.5$  has an inherent *genetic drift* towards the optimum ( $n/2$ ). Therefore the case  $\alpha = \beta = 0.5$  should be easy to optimise even for a  $(1, \lambda)$  CoEA that selects a random offspring every time.

For the three different search spaces the  $(1, \lambda)$  CoEA uses the following mutation operators:

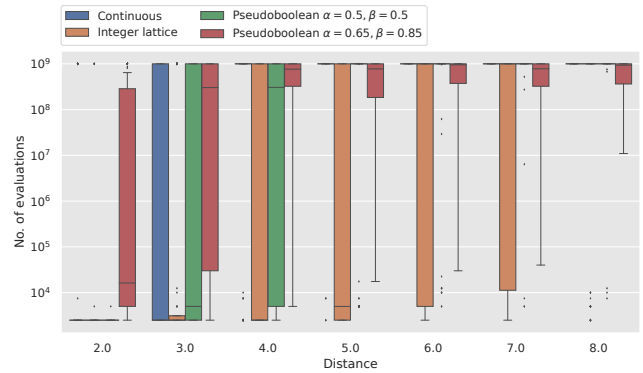
- $\mathcal{X} = \mathcal{Y} = \mathbb{Z}$  - **Geometric mutation** (random direction and  $p = 1/2$ ): For a parent  $x$ , the offspring  $x' = x + \mathcal{G}(1/2)$  with probability  $1/2$  and  $x' = x - \mathcal{G}(1/2)$ .
- $\mathcal{X} = \mathcal{Y} = \mathbb{R}$  - **Gaussian mutation** (step size of 1): For a parent  $x$ , the offspring  $x' = x + \mathcal{N}(0, 1)$ .
- $\mathcal{X} = \mathcal{Y} = \{0, 1\}^n$  - **Standard bit mutation** ( $p = 1/n$ ): An offspring is created by copying the parent and flipping each bit with probability  $1/n$ .

All experiments comprise of 100 runs for each algorithm-problem pair, recording the number of evaluations (interactions) to reach the optimum. Each run is limited to  $10^9$  evaluations, if the run does not find the optimum within this time limit,  $10^9$  is reported as the runtime. For the search space  $\mathcal{X} = \mathcal{Y} = \mathbb{R}$  we assume the optimum is found if both  $x = [\beta - 0.1, \beta + 0.1]$  and  $y = [\alpha - 0.1, \alpha + 0.1]$ .

We first explore the fitness measure  $f^{\text{wrs}}$ . In Figure 3 we show in the  $x$ -axis the initial Manhattan distance to the optimum<sup>3</sup> and in the  $y$ -axis the runtime divided by  $d^2$  and log-scaled. Given that the number of evaluations are normalised by  $d^2$  we can appreciate that for the distances studied and all search spaces, the runtime seems to grow slower than  $d^2$ . This indicates that our bounds might not be tight or that they are tight but only for large initial distances  $d$ . Nonetheless, these experiments are evidence that our theoretical results on  $f^{\text{wrs}}$  do translate to other search spaces.

Similarly, the experimental results for the fitness measure  $f^{\text{avg}}$  (Figure 4) show that our theoretical results translate to other search spaces. For these experiments it is particularly interesting to see

<sup>3</sup>For the search space  $\mathcal{X} = \mathcal{Y} = \{0, 1\}^n$  the Manhattan distance is defined as the sum of the Hamming distances of both  $x$  and  $y$  to an optimal solution.



**Figure 4: Runtime of the  $(1, \lambda)$  CoEA (Algorithm 1) with fitness measure  $f^{\text{avg}}$  on BILINEAR.**

that the runtime *explodes* even for small initial distances to the optimum.

## 8 CONCLUSIONS

We have shown how the selection of fitness measures dramatically affects the performance of the  $(1, \lambda)$  CoEA on BILINEAR. Both fitness measures studied here present a cyclic behaviour on this problem. Using the average payoff of interactions worsens this behaviour resulting in exponential time to find an optimal solution. On the other hand using the worst payoff alleviates the problem resulting in an efficient optimisation.

An important insight from our theoretical analysis is that both fitness measures tend to maintain the distance to the optimum in most generations and what differentiates them is the behaviour during the generation where the current solutions are part of more than one quadrant of the search space. While using the worst interaction as fitness decreases the distance to the optimum averaging interactions increases it. We hope that this insight inspires the design of better CoEAs that exploit this behaviour.

It remains an open problem, whether the results shown here apply to other intransitive problems and how other fitness measures affect the performance of CoEAs on intransitive problems. Furthermore, an interesting venue for future work is to theoretically analyse other classes of problems to examine whether CoEAs are affected by the fitness measure on these problems to the same extent as seen in this work.

## ACKNOWLEDGMENTS

This research was supported by a Turing AI Fellowship (EPSRC grant ref EP/V025562/1). The computations described in this paper were performed using the University of Birmingham’s BlueBEAR HPC service. See <http://www.birmingham.ac.uk/bear> for more details.

## REFERENCES

- [1] Jürgen Branke and Johanna Rosenbusch. New approaches to coevolutionary worst-case optimization. In *Parallel Problem Solving from Nature – PPSN X*, pages 144–153, Berlin, Heidelberg, 2008. Springer Berlin Heidelberg.
- [2] Anthony Bucci. *Emergent Geometric Organization and Informative Dimensions in Coevolutionary Algorithms*. Brandeis University, Massachusetts, USA, 2007.



- [3] Benjamin Doerr. *Probabilistic Tools for the Analysis of Randomized Optimization Heuristics*, pages 1–87. Springer, 2020.
- [4] Diana Flores, Erik Hemberg, Jamal Toutouh, and Una-May O'Reilly. Coevolutionary generative adversarial networks for medical image augmentation at scale. In *Proceedings of the Genetic and Evolutionary Computation Conference, GECCO '22*, page 367–376, New York, NY, USA, 2022. Association for Computing Machinery. ISBN 9781450392372.
- [5] Erik Hemberg, Jamal Toutouh, Abdullah Al-Dujaili, Tom Schmiedlechner, and Una-May O'Reilly. Spatial coevolution for generative adversarial network training. *ACM Trans. Evol. Learn. Optim.*, 1(2), jul 2021. ISSN 2688-299X.
- [6] W.Daniel Hillis. Co-evolving parasites improve simulated evolution as an optimization procedure. *Physica D: Nonlinear Phenomena*, 42(1-3):228–234, June 1990.
- [7] Thomas Jansen and R. Paul Wiegand. The Cooperative Coevolutionary (1+1) EA. *Evolutionary Computation*, 12(4):405–434, 12 2004.
- [8] Mikkel T Jensen. A New Look at Solving Minimax Problems with Coevolution. *Metaheuristics: Computer Decision-Making*, 86:369, 2003.
- [9] Per Kristian Lehre. Runtime analysis of competitive co-evolutionary algorithms for maximin optimisation of a bilinear function. In *Proceedings of the Genetic and Evolutionary Computation Conference, GECCO '22*, page 1408–1416, New York, NY, USA, 2022. Association for Computing Machinery.
- [10] Johannes Lengler. *Drift Analysis*, pages 89–131. Springer, 2020.
- [11] Tengyuan Liang and James Stokes. Interaction Matters: A Note on Non-asymptotic Local Convergence of Generative Adversarial Networks. *ArXiv e-prints*, 2019.
- [12] Sean Luke and R Paul Wiegand. When coevolutionary algorithms exhibit evolutionary dynamics. In *Genetic and Evolutionary Computation Conference Workshop Program*, pages 236–241, 2002.
- [13] Xiaoliang Ma, Xiaodong Li, Qingfu Zhang, Ke Tang, Zhengping Liang, Weixin Xie, and Zexuan Zhu. A survey on cooperative co-evolutionary algorithms. *IEEE Transactions on Evolutionary Computation*, 23(3):421–441, 2019. doi: 10.1109/TEVC.2018.2868770.
- [14] Aryan Mokhtari, Asuman Ozdaglar, and Sarath Pattathil. A Unified Analysis of Extra-gradient and Optimistic Gradient Methods for Saddle Point Problems: Proximal Point Approach. *ArXiv e-prints*, 2019.
- [15] Pietro S. Oliveto and Carsten Witt. Improved time complexity analysis of the simple genetic algorithm. *Theoretical Computer Science*, 605:21 – 41, 2015. ISSN 0304-3975.
- [16] Elena Popovici, Anthony Bucci, R. Paul Wiegand, and Edwin D. De Jong. Coevolutionary Principles. In Grzegorz Rozenberg, Thomas Bäck, and Joost N. Kok, editors, *Handbook of Natural Computing*, pages 987–1033. Springer Berlin Heidelberg, Berlin, Heidelberg, 2012.
- [17] Christopher D. Rosin and Richard K. Belew. New Methods for Competitive Coevolution. *Evolutionary Computation*, 5(1):1–29, 03 1997. ISSN 1063-6560.
- [18] Karl Sims. Evolving 3d morphology and behavior by competition. *Artificial Life*, 1(4):353–372, 1994.
- [19] Martin J. Wainwright. *Basic tail and concentration bounds*, page 21–57. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2019. doi: 10.1017/9781108627771.002.
- [20] R. Paul Wiegand, William C. Liles, and Kenneth A. De Jong. An empirical analysis of collaboration methods in cooperative coevolutionary algorithms. In *Proceedings of the 3rd Annual Conference on Genetic and Evolutionary Computation, GECCO '01*, page 1235–1242, San Francisco, CA, USA, 2001. Morgan Kaufmann Publishers Inc. ISBN 1558607749.
- [21] Rudolf Paul Wiegand. *An analysis of cooperative coevolutionary algorithms*. George Mason University, Virginia, USA, 2004.
- [22] Guojun Zhang and Yaoliang Yu. Convergence of Gradient Methods on Bilinear Zero-Sum Games. *ArXiv e-prints*, 2020.

## A FULL LEMMA 3.2

**Lemma A.1** (Fitness Based on Worst Interactions). *Consider two populations of solutions  $P_t$  and  $Q_t$  and the fitness measure  $f^{\text{WTS}}$  on BILINEAR. Then  $f^{\text{WTS}}$  assigns the highest fitness to all solutions in the sets  $\xi \subseteq P_t, \Upsilon \subseteq Q_t$  defined as,*

$$\xi = \begin{cases} \{x \mid x \in P_t \wedge x = x_{\max}^- \} & \text{if } |Y_t| \neq \bar{Y}_t \wedge |X_t| \neq \bar{X}_t \wedge (\beta - x_{\max}^-)(\alpha - y_{\max}) \geq (\beta - x_{\min}^+)(\alpha - y_{\min}), & (17) \\ \{x \mid x \in P_t \wedge x = x_{\min}^+ \} & \text{if } |Y_t| \neq \bar{Y}_t \wedge |X_t| \neq \bar{X}_t \wedge (\beta - x_{\max}^-)(\alpha - y_{\max}) \leq (\beta - x_{\min}^+)(\alpha - y_{\min}), & (18) \\ \{x \mid x \in P_t \wedge x = x_{\min}^* \} & \text{if } |Y_t| \neq \bar{Y}_t \wedge |X_t| = \bar{X}_t, & (19) \\ P_t^+ & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t > 0 \wedge \exists y \in Q_t, y = \alpha \wedge P_t^+ \neq \emptyset, & (20) \\ \{x \mid x \in P_t \wedge x = x_{\min}^* \} & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t > 0 \wedge \exists y \in Q_t, y = \alpha \wedge P_t^+ = \emptyset, & (21) \\ P_t^- & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t < 0 \wedge \exists y \in Q_t, y = \alpha \wedge P_t^- \neq \emptyset, & (22) \\ \{x \mid x \in P_t \wedge x = x_{\min}^* \} & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t < 0 \wedge \exists y \in Q_t, y = \alpha \wedge P_t^- = \emptyset, & (23) \\ \{x \mid x \in P_t \wedge x = x_{\max} \} & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t > 0 \wedge \forall y \in Q_t, y \neq \alpha, & (24) \\ \{x \mid x \in P_t \wedge x = x_{\min} \} & \text{if } |Y_t| = \bar{Y}_t \wedge Y_t < 0 \wedge \forall y \in Q_t, y \neq \alpha, & (25) \\ P_t & \text{if } y = \alpha \forall y \in Q_t, & (26) \end{cases}$$

and

$$\Upsilon = \begin{cases} \{y \mid y \in Q_t \wedge y = y_{\max}^- \} & \text{if } |X_t| \neq \bar{X}_t \wedge |Y_t| \neq \bar{Y}_t \wedge (\alpha - y_{\max}^-)(\beta - x_{\min}) < (y_{\min}^+ - \alpha)(x_{\max} - \beta), & (27) \\ \{y \mid y \in Q_t \wedge y = y_{\min}^+ \} & \text{if } |X_t| \neq \bar{X}_t \wedge |Y_t| \neq \bar{Y}_t \wedge (\alpha - y_{\max}^-)(\beta - x_{\min}) > (y_{\min}^+ - \alpha)(x_{\max} - \beta), & (28) \\ \{y \mid y \in Q_t \wedge y = y_{\min}^* \} & \text{if } |X_t| \neq \bar{X}_t \wedge |Y_t| = \bar{Y}_t, & (29) \\ Q_t^+ & \text{if } |X_t| = \bar{X}_t \wedge X_t > 0 \wedge \exists x \in P_t, x = \beta \wedge Q_t^+ \neq \emptyset, & (30) \\ \{y \mid y \in Q_t \wedge y = y_{\min}^* \} & \text{if } |X_t| = \bar{X}_t \wedge X_t > 0 \wedge \exists x \in P_t, x = \beta \wedge Q_t^+ = \emptyset, & (31) \\ Q_t^- & \text{if } |X_t| = \bar{X}_t \wedge X_t < 0 \wedge \exists x \in P_t, x = \beta \wedge Q_t^- \neq \emptyset, & (32) \\ \{y \mid y \in Q_t \wedge y = y_{\min}^* \} & \text{if } |X_t| = \bar{X}_t \wedge X_t < 0 \wedge \exists x \in P_t, x = \beta \wedge Q_t^- = \emptyset, & (33) \\ \{y \mid y \in Q_t \wedge y = y_{\max} \} & \text{if } |X_t| = \bar{X}_t \wedge X_t > 0 \wedge \forall x \in Q_t, x \neq \beta, & (34) \\ \{y \mid y \in Q_t \wedge y = y_{\min} \} & \text{if } |X_t| = \bar{X}_t \wedge X_t < 0 \wedge \forall x \in P_t, x \neq \beta, & (35) \\ Q_t & \text{if } x = \beta \forall x \in P_t, & (36) \end{cases}$$

respectively.

**PROOF.** In this proof we show what is the individual  $x$  with highest fitness and note that the same arguments (with slight variations) result in the individual  $y$  with highest fitness. Given that there are several cases we go through each of them in the order that they appear in the statement of Lemma A.1.

For simplicity, let  $x_i := P_t(i)$  and  $y_i := Q_t(i)$ , then

$$f_X^{\text{WTS}}(x_i, Q_t) = \min_{i \in [\lambda]} \{g(x_i, y_i)\} = \min_{y \in Q_t} \{x_i y_i - \alpha x_i - \beta y_i\}$$

We note that the condition  $|X_t| \neq \bar{X}_t$  implies that there are individuals in population  $P_t$  with both  $x_i < \beta$  and  $x_i > \beta$  and  $|X_t| = \bar{X}_t$  implies that all individuals in population  $P_t$  have either  $x_i \leq \beta$  or  $x_i \geq \beta$ . The same relations apply for the population  $Q_t$ .

Due to the note above Conditions (27) and (28) imply that the pairs of individuals in the Cartesian product  $P_t \times Q_t$  lie in the four quadrants of the search space. Then, for any  $x_i < \alpha$  the minimum value is attained for the  $y$  with the highest value, for any  $x > \alpha$  the minimum value is attained for the  $y$  with the lowest value.

If we only consider  $x_i \leq \alpha$  the fitness of every individual is

$$f_X^{\text{WTS}}(x_i, Q_t) = x_i y_{\max} - \alpha x_i - \beta y_{\max}.$$

and the individual with highest fitness is the individual nearest to  $\beta$  with  $x_i \leq \alpha$ , that is  $x_{\max}^-$ . Therefore its fitness is

$$f_X^{\text{WTS}}(x_{\max}^-, Q_t) = x_{\max}^- y_{\max} - \alpha x_{\max}^- - \beta y_{\max}. \quad (37)$$

If we only consider  $x_i \geq \alpha$  the fitness of every individual is

$$f_X^{\text{WTS}}(x_i, Q_t) = x_i y_{\min} - \alpha x_i - \beta y_{\min}. \quad (38)$$

and the individual with highest fitness is the individual nearest to  $\beta$  with  $x_i \geq \alpha$ , that is  $x_{\min}^+$ . Therefore its fitness is

$$f_X^{\text{WTS}}(x_{\min}^+, Q_t) = x_{\min}^+ y_{\min} - \alpha x_{\min}^+ - \beta y_{\min}.$$

Conditions (27) and (28) also rely on the following values:

$$\begin{aligned}(\beta - x_{\max}^-)(\alpha - y_{\max}) &= \alpha\beta + x_{\max}^- y_{\max} - \alpha x_{\max}^- - \beta y_{\max} \\(\beta - x_{\min}^+)(\alpha - y_{\min}) &= \alpha\beta + x_{\min}^+ y_{\max} - \alpha x_{\min}^+ - \beta y_{\min}.\end{aligned}$$

Then,

$$\begin{aligned}(\beta - x_{\max}^-)(\alpha - y_{\max}) &\geq (\beta - x_{\min}^+)(\alpha - y_{\min}) \\ \Leftrightarrow f_{\mathcal{X}}^{\text{WTS}}(x_{\max}^-, Q_t) &\geq f_{\mathcal{X}}^{\text{WTS}}(x_{\min}^+, Q_t),\end{aligned}$$

and  $x_{\max}^-$  has the highest fitness in the population. If the inequality is in the other direction  $x_{\min}^+$  has the highest fitness. This proves the first two cases.

The Condition (19) imply that the pairs of individuals in the Cartesian product  $P_t \times Q_t$  lie in the two left or right quadrants of the search space. Due to Equations (37) and (38) the highest individual is the closest to  $\beta$  showing the third case.

The Conditions (20) and (21) imply that the pairs of individuals in the Cartesian product  $P_t \times Q_t$  lie in the top part of the search space and there is an optimal individual in  $Q_t$ . Additionally if  $P_t^+ = \emptyset$  the pairs of individuals only lie in the first quadrant ( $x_i < \beta$ ) then the minimum payoff of all individuals is attained with  $y_{\max}$  and as in Equation (37) the highest fitness is attained for  $x_{\max}^-$ . If  $P_t^+ \neq \emptyset$  then there are some pairs of individuals in the second quadrant ( $x_i \geq \beta$ ). In this case the fitness of individuals in the second quadrant is higher than the ones in the first quadrant and their minimum payoff is attained for  $y = \alpha$  giving them all the same fitness,

$$f_{\mathcal{X}}^{\text{WTS}}(x_i, Q_t) = \min_{y \in Q_t} \{x_i y_i - \alpha x_i - \beta y_i\} = x_i \alpha - \alpha x_i - \beta \alpha = -\beta \alpha.$$

This proves the fourth and fifth case. For the sixth and seventh case the arguments are the same but mirrored for the bottom part of the search space.

Conditions (24) and (25) imply that the pairs of individuals in the Cartesian product  $P_t \times Q_t$  lie in exactly one quadrant and there are no optimal individuals in  $Q_t$ . The Condition (24) refers to the first and second quadrant and Condition (25) refers to the third and fourth quadrant.

For Condition (24) since  $y_i > \alpha$  the fitness of all individuals is

$$f_{\mathcal{X}}^{\text{WTS}}(x_i, Q_t) = \min_{y \in Q_t} \{x_i y_i - \alpha x_i - \beta y_i\} = x_i y_{\max} - \alpha x_i - \beta y_{\max} = x_i (y_{\max} - \alpha) - \beta y_{\max},$$

therefore, the individual  $x$  with highest value has the highest fitness. For Condition (25)  $y_i < \alpha$  and similarly the fitness of all individuals is

$$f_{\mathcal{X}}^{\text{WTS}}(x_i, Q_t) = \min_{y \in Q_t} \{x_i y_i - \alpha x_i - \beta y_i\} = x_i y_{\min} - \alpha x_i - \beta y_{\min} = x_i (y_{\min} - \alpha) - \beta y_{\min}.$$

Since  $(y_{\min} - \alpha) < 0$ , the individual  $x$  with lowest value has the highest fitness.

Finally, the last case says that all  $y \in Q_t$  have  $y = \alpha$ , therefore all individuals in  $P_t$  have the same fitness

$$f_{\mathcal{X}}^{\text{WTS}}(x_i, Q_t) = \min_{y \in Q_t} \{x_i y_i - \alpha x_i - \beta y_i\} = x_i \alpha - \alpha x_i - \beta \alpha = -\beta \alpha.$$

□

## B OMITTED PROOFS

This appendix contains the proofs omitted from the main part.

**Lemma 3.1** (Fitness Based on Averaging Interactions). *Consider two populations of solutions  $P_t$  and  $Q_t$  and the fitness measure  $f^{\text{avg}}$  on BILINEAR. Then  $f^{\text{avg}}$  assigns the highest fitness to all solutions in the sets  $\xi \subseteq P_t, \Upsilon \subseteq Q_t$  defined as,*

$$\xi = \begin{cases} \{x \mid x \in P_t \wedge x = x_{\min}\} & \text{if } Y_t < 0, \\ \{x \mid x \in P_t \wedge x = x_{\max}\} & \text{if } Y_t > 0, \\ P_t & \text{otherwise,} \end{cases}$$

and

$$\Upsilon = \begin{cases} \{y \mid y \in Q_t \wedge y = y_{\max}\} & \text{if } X_t < 0, \\ \{y \mid y \in Q_t \wedge y = y_{\min}\} & \text{if } X_t > 0, \\ Q_t & \text{otherwise.} \end{cases}$$

**PROOF OF LEMMA 3.1.** We use the fitness measure  $f^{\text{avg}}$  to assign a value to each individual. For every individual  $x \in P_t$  its fitness measure is

$$\begin{aligned} f(x, Q_t) &= \frac{1}{\lambda} \sum_{y \in Q_t} xy - \alpha x - \beta y \\ &= \frac{1}{\lambda} \sum_{y \in Q_t} x(y - \alpha) - \sum_{y \in Q_t} \beta y. \end{aligned}$$

In the same manner, for every individual  $y \in Q_t$  its fitness measure is

$$\begin{aligned} f(y, P_t) &= -\frac{1}{\lambda} \sum_{x \in P_t} xy - \alpha x - \beta y \\ &= -\frac{1}{\lambda} \sum_{x \in P_t} y(x - \beta) - \sum_{x \in P_t} \alpha x. \end{aligned}$$

Since we are only comparing their fitness, if we add, subtract or multiply a positive number to all the fitness values the comparison stays equal. Therefore, we can change the fitness measures for

$$\begin{aligned} f(x, Q_t) &= x \sum_{y \in Q_t} (y - \alpha) =: xY_t \text{ and} \\ f(y, P_t) &= -y \sum_{x \in P_t} (x - \beta) =: -yX_t, \end{aligned}$$

without changing the results of these comparisons. With this result we can see that if  $Y_t > 0$  the individual  $x$  with the largest value has the highest fitness. Similarly, if  $Y_t < 0$  the individual with the smallest has the highest fitness. And, if  $Y_t = 0$  all individuals  $x$  have the same fitness. Using the same arguments and noting that the sign in  $f(y, P_t)$  reverse the order of fitness yields the statement for  $y$ .  $\square$

**Lemma 4.1.** *Let  $h \in \mathbb{N}$  and  $Y_i$  be i.i.d. random variables sampled from  $\mathcal{G}(p)$ . Then,*

$$\Pr \left[ \sum_{i=1}^{\psi} (Y_i + h) - \sum_{i=\psi+1}^{2\psi} (Y_i - h) \leq 0 \right] = e^{-\Omega(h\psi)}.$$

Before showing Lemma 4.1 we need the following helper lemma.

**Lemma B.1.** *Let  $0 < p < 1$  be a constant,  $h \in \mathbb{N}$ ,  $\psi \in \mathbb{N}$ ,  $X_i \sim \mathcal{G}(p)$  and  $X := \sum_{i=1}^{\psi} X_i$  then, there exists a constant  $c < 1$  for which the following statements are simultaneously true*

$$\begin{aligned} \Pr \left[ X \leq \left( \frac{c(1-p)}{p} - h \right) \psi \right] &\leq e^{-\Omega(h\psi)} \\ \Pr \left[ X \geq \left( \frac{c(1-p)}{p} + h \right) \psi \right] &\leq e^{-\Omega(h\psi)}. \end{aligned}$$

**PROOF OF LEMMA B.1.** If  $h > \frac{c(1-p)}{p}$  then  $\left( \frac{c(1-p)}{p} - h \right) < 0$  and the first probability is 0. Otherwise, we note that for  $\delta := 1 - c + \frac{hp}{1-p}$

$$\Pr \left[ X \leq \left( \frac{c(1-p)}{p} - h \right) \psi \right] = \Pr \left[ X \leq (1 - \delta) \frac{1-p}{p} \psi \right].$$

Then, using Chernoff bounds [3, Theorem 1.10.32] we obtain

$$\Pr \left[ X \leq (1 - \delta) \frac{(1-p)}{p} \psi \right] \leq \exp \left( -\frac{\delta^2 \psi}{2 - \frac{4\delta}{3}} \right) \leq \exp \left( -\frac{\delta^2 \psi}{2} \right) \leq \exp \left( -\left( \frac{hp}{1-p} \right)^2 \frac{\psi}{2} \right) = \exp \left( -\frac{h^2 p^2 \psi}{2(1-p)^2} \right) = e^{-\Omega(h\psi)}.$$

The last inequality holds for all constant  $0 \leq c \leq 1$ .

For the second statement, let  $\delta := \frac{hp}{1-p} - 1 + c$ , then for all  $c > \frac{1-p}{1-p}$  and all  $h \in \mathbb{N}$  it holds that  $\delta > 0$  and

$$\begin{aligned} \Pr \left[ X \geq \left( \frac{c(1-p)}{p} + h \right) \psi \right] &= \Pr \left[ X \geq \left( \frac{hp}{1-p} + c \right) \frac{1-p}{p} \psi \right] \\ &= \Pr \left[ X \geq (1 + \delta) \frac{1-p}{p} \psi \right]. \end{aligned}$$

Finally, using Chernoff bounds [3, Theorem 1.10.32] we obtain

$$\Pr \left[ X \geq (1 + \delta) \frac{1-p}{p} \psi \right] \leq \exp \left( -\frac{\delta^2(\psi-1)}{2(1+\delta)} \right) = e^{-\Omega(h\psi)}.$$

Noting that for all constant  $0 < p < 1$ ,  $\frac{1-2p}{1-p} < 1$  shows that there exists a  $c < 1$  for which the above holds.  $\square$

With Lemma B.1 we can prove Lemma 4.1.

PROOF OF LEMMA 4.1. Let

$$Y := \sum_{i=1}^{\psi} (Y_i + h) - \sum_{i=\psi+1}^{2\psi} (Y_i - h) = 2\psi h + \sum_{i=1}^{\psi} Y_i - \sum_{i=\psi+1}^{2\psi} Y_i.$$

We pessimistically assume that if  $\sum_{i=1}^{\psi} Y_i \leq \left( \frac{c(1-p)}{p} - h \right) \psi$  (for some  $c < 1$  that meets the condition of Lemma B.1) then  $Y \leq 0$ . By Lemma B.1 the probability of this event is at most  $e^{-\Omega(h\psi)}$ . Otherwise,

$$2\psi h + \sum_{i=1}^{\psi} Y_i - \sum_{i=\psi+1}^{2\psi} Y_i \geq \psi \left( \frac{c(1-p)}{p} + h \right) + 1 - \sum_{i=\psi+1}^{2\psi} Y_i.$$

By Lemma B.1 the probability that  $\sum_{i=\psi+1}^{2\psi} Y_i \geq \psi \left( \frac{c(1-p)}{p} + h \right)$  is at most  $e^{-\Omega(h\psi)}$ . A union bound of both events shows that  $\Pr [Y \leq 0] = e^{-\Omega(h\psi)}$ .  $\square$

**Lemma 4.2.** Let  $0 < p < 1$ ,  $\psi \in \mathbb{N}$ ,  $X_i \sim \mathcal{G}(p)$  and  $X^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$  then

$$\left(1 - e^{-\frac{1}{1-p}}\right) \log_{\frac{1}{1-p}} \psi \leq \mathbb{E} \left[ X^{(\psi)} \right] \leq \log_{\frac{1}{1-p}} \psi + \frac{1-p}{p} \quad (11)$$

PROOF OF LEMMA 4.2. The random variable  $X^{(\psi)}$  is non-negative, therefore we can compute its expectation by  $\mathbb{E} \left[ X^{(\psi)} \right] = \sum_{k=0}^{\infty} \Pr \left[ X^{(\psi)} \geq k \right]$ . Since  $X^{(\psi)}$  is the maximum of  $\psi$  random variables the  $\Pr \left[ X^{(\psi)} \geq k \right] = 1 - \Pr [X_i < k]^\psi = 1 - (1 - \Pr [X_i \geq k])^\psi$ . Given that the  $X_i$  random variables are sampled from a geometric distribution with probability  $p$  the  $\Pr [X_i \geq k] = (1-p)^k$ . Therefore,

$$\mathbb{E} \left[ X^{(\psi)} \right] = \sum_{k=0}^{\infty} 1 - (1 - (1-p)^k)^\psi.$$

For the lower bound we only take into account the first  $\log_{1/(1-p)} \psi$  summands.

$$\begin{aligned} \mathbb{E} \left[ X^{(\psi)} \right] &\geq \sum_{k=0}^{\log_{\frac{1}{1-p}} \psi - 1} 1 - (1 - (1-p)^k)^\psi \\ &\geq \sum_{k=0}^{\log_{\frac{1}{1-p}} \psi - 1} 1 - \left( 1 - \frac{(1-p)^{\log_{\frac{1}{1-p}} \psi}}{1-p} \right)^\psi \\ &\geq \sum_{k=0}^{\log_{\frac{1}{1-p}} \psi - 1} 1 - \left( 1 - \frac{1}{\psi} \right)^\psi \\ &\geq \left( 1 - e^{-\frac{1}{1-p}} \right) \log_{\frac{1}{1-p}} \psi. \end{aligned}$$

For the upper bound if  $\log_{1/(1-p)} \psi \geq 0$  we assume that the first  $\log_{1/(1-p)} \psi$  summands are 1 and compute the rest as follows.

$$\begin{aligned}
 \mathbb{E}[X^{(\psi)}] &\leq \log_{\frac{1}{1-p}} \psi + \sum_{k=\log_{\frac{1}{1-p}} \psi}^{\infty} 1 - (1 - (1-p)^k)^\psi \\
 &\leq \log_{\frac{1}{1-p}} \psi + \sum_{k=0}^{\infty} 1 - \left(1 - (1-p)^k (1-p)^{\log_{\frac{1}{1-p}} \psi}\right)^\psi \\
 &= \log_{\frac{1}{1-p}} \psi + \sum_{k=0}^{\infty} 1 - \left(1 - \frac{(1-p)^k}{\psi}\right)^\psi \\
 &\leq \log_{\frac{1}{1-p}} \psi + \sum_{k=0}^{\infty} (1-p)^k \\
 &= \log_{\frac{1}{1-p}} \psi + \frac{1}{p}
 \end{aligned}$$

□

**Lemma 4.3.** *Let  $0 < p < 1$  and  $c > 0$ . Let  $\delta > 0$ ,  $\psi \in \mathbb{N}$ ,  $X_i \sim \mathcal{G}(p)$  and  $X^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$  then*

$$\Pr \left[ X^{(\psi)} \geq (1 + \delta) \frac{\ln \psi}{p} \right] \leq \min \{ \psi^{-\delta}, 1 \} \quad (12)$$

$$\Pr \left[ X^{(\psi)} \geq \log_{\frac{1}{1-p}} c \psi \right] \leq 1/c \quad (13)$$

$$\Pr \left[ X^{(\psi)} < (1 - \delta) \frac{p \ln \psi}{1-p} \right] \leq \exp(-\psi^\delta) \quad (14)$$

**PROOF OF LEMMA 4.3.** Given that  $X_i \sim \mathcal{G}(p)$  the  $\Pr[X_i] \geq k$  is the probability that the first  $k$  Bernoulli trials are unsuccessful, that is  $\Pr[X_i] \geq k = (1-p)^k$ . Then we have

$$\begin{aligned}
 \Pr[X \geq k] &= 1 - \Pr[X_i < k]^\psi \\
 &= 1 - (1 - \Pr[X_i \geq k])^\psi \\
 &= 1 - (1 - (1-p)^k)^\psi
 \end{aligned} \quad (39)$$

Using  $e^x \geq 1 + x$  and  $(1+x)^r \geq 1 + rx$  for  $x \geq -1$ ,  $r \in \mathbb{R} \setminus (0, 1)$  we have

$$\Pr[X \geq k] \leq 1 - (1 - e^{-pk})^\psi \leq \psi e^{-pk}$$

To show Equation (12), we define  $k := (1 + \delta) \frac{\ln \psi}{p}$  obtaining

$$\Pr[X \geq k] \leq \psi e^{-(1+\delta) \ln \psi} = \psi^{-\delta}$$

For Equation (13), let  $k = \log_{1/(1-p)} c \psi$ , then Equation (39) yields,

$$\Pr[X \geq k] = 1 - \left(1 - (1-p)^{\log_{\frac{1}{1-p}} c \psi}\right)^\psi = 1 - \left(1 - \frac{1}{c}\right)^\psi \quad (40)$$

Using  $(1 + \frac{x}{n})^n \geq 1 + x$  yields the second statement.

Finally, to show Equation (14), we define  $k := (1 - \delta) \frac{p \ln \psi}{1-p}$  and use Equation (39) as follows

$$\begin{aligned} \Pr[X < k] &= 1 - \Pr[X \geq k] = (1 - (1-p)^k)^\psi \\ &= \left(1 - (1-p)^{(1-\delta) \frac{p \ln \psi}{1-p}}\right)^\psi \\ &= \left(1 - \left((1-p)^{\frac{p}{1-p}}\right)^{(1-\delta) \ln \psi}\right)^\psi \\ &\leq \left(1 - e^{-(1-\delta) \ln \psi}\right)^\psi = \left(1 - \frac{1}{\psi^{1-\delta}}\right)^\psi \\ &= \left(\left(1 - \frac{1}{\psi^{1-\delta}}\right)^{\psi^{1-\delta}}\right)^{\psi^\delta} \leq e^{-\psi^\delta} \end{aligned}$$

□

**Lemma 4.4.** Let  $0 < p < 1$ . Let  $\psi \in \mathbb{N}$ ,  $\tau \in \mathbb{N}$ ,  $X_i, Y_i$  be independent and identically distributed (i.i.d.) random variables sampled from  $\mathcal{G}(p)$ ,  $X_j^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$  and  $Y_j^{(\psi)} := \max_{i \in [\psi]} \{Y_i\}$ . Let  $Z := \sum_{j=1}^{\tau} (X_j^{(\psi)} - Y_j^{(\psi)})$ . Then  $\mathbb{E}[Z] = 0$ .

**PROOF OF LEMMA 4.4.** Given that  $X_i$  and  $Y_i$  are i.i.d. random variables, then  $X_j^{(\psi)}$  and  $Y_j^{(\psi)}$  are also i.i.d. and  $\mathbb{E}[X_j^{(\psi)}] = \mathbb{E}[Y_j^{(\psi)}]$ . Therefore,

$$\mathbb{E}[Z] = \mathbb{E}\left[\sum_{j=1}^{\tau} X_j^{(\psi)} - Y_j^{(\psi)}\right] = \sum_{i=1}^{\tau} \mathbb{E}[X_i^{(\psi)}] - \mathbb{E}[Y_i^{(\psi)}] = 0.$$

□

**Lemma 4.5.** Let  $0 < \varepsilon < p < 1$ . Let  $|\eta| < p - \varepsilon$ ,  $\psi \in \mathbb{N}$ ,  $\tau \in \mathbb{N}$ ,  $X_i \sim \mathcal{G}(p)$  and  $X^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$ . Then  $\mathbb{E}[e^{\eta X^{(\psi)}}] \leq \frac{\psi}{\varepsilon}$  and  $\mathbb{E}[e^{-\eta X^{(\psi)}}] \leq \psi \left(1 + \frac{1}{\varepsilon}\right)$ .

**PROOF OF LEMMA 4.5.** By definition,

$$\mathbb{E}[e^{\eta X^{(\psi)}}] = \sum_{j=0}^{\infty} e^{\eta j} \Pr[X^{(\psi)} = j] \leq \sum_{j=0}^{\infty} e^{\eta j} \Pr[X^{(\psi)} \geq j].$$

By a union bound,

$$\mathbb{E}[e^{\eta X^{(\psi)}}] \leq \sum_{j=0}^{\infty} e^{\eta j} \sum_{i=1}^{\psi} \Pr[X_i \geq j] = \psi \sum_{j=0}^{\infty} e^{\eta j} (1-p)^j.$$

Finally using the assumption  $|\eta| < p - \varepsilon$  and  $(1+x) \leq e^x$  for all  $x \in \mathbb{R}$  we obtain,

$$\begin{aligned} \mathbb{E}[e^{\eta X^{(\psi)}}] &\leq \psi \sum_{j=0}^{\infty} e^{\eta j} e^{-pj} \leq \psi \sum_{j=0}^{\infty} e^{-\varepsilon j} \\ &= \frac{\psi e^\varepsilon}{e^\varepsilon - 1} = \psi \left(1 + \frac{1}{e^\varepsilon - 1}\right) \leq \psi \left(1 + \frac{1}{\varepsilon}\right). \end{aligned}$$

□

**Lemma 4.6.** Let  $0 < p < 1$ . Let  $\psi^6 \geq \left(1 + \frac{2}{p}\right)^2$  with  $\psi \in \mathbb{N}$ . Let  $r' := \frac{8r\sqrt{2 \ln \psi}}{p}$ ,  $r \geq 2$ ,  $j \in \mathbb{N}_0$ ,  $\tau \in \mathbb{N}$ ,  $X_i, Y_i \sim \mathcal{G}(p)$  be i.i.d. random variables,  $X^{(\psi)} := \max_{i \in [\psi]} \{X_i\}$  and  $Y^{(\psi)} := \max_{i \in [\psi]} \{Y_i\}$ . Let  $Z := \sum_{i=1}^{\tau} (X^{(\psi)} - Y^{(\psi)})$ . Then,

$$\Pr[|Z| \geq jr'] \leq e^{-jr'}.$$

**PROOF OF LEMMA 4.6.** The case with  $j = 0$  is trivially true, from now on we only consider  $j \in \mathbb{N}$ . The main proof idea is to show that  $Z$  is sub-exponential with parameters  $(v, b)$ . That is, we want to show that there exists a  $v > 0$  such that  $\mathbb{E}[e^{\eta(X^{(\psi)} - Y^{(\psi)})}] \leq e^{\frac{v^2 \eta^2}{2}}$  for all

$|\eta| < 1/b$  and use known tail bounds for the sum of sub-exponential random variables [19]. Lemma 4.5 with  $\varepsilon = p/2$  and the assumption  $\psi^6 \geq \left(1 + \frac{2}{p}\right)^2$  yield,

$$\begin{aligned} \mathbb{E}\left[e^{\eta(X^{(\psi)} - Y^{(\psi)})}\right] &= \mathbb{E}\left[e^{\eta X^{(\psi)}}\right] \mathbb{E}\left[e^{-\eta Y^{(\psi)}}\right] \\ &\leq \psi^2 \left(1 + \frac{2}{p}\right)^2 \leq \psi^8 = e^{8 \ln \psi}. \end{aligned}$$

Let  $v := \frac{8\sqrt{\ln \psi}}{p}$  and recall that  $\varepsilon = p/2$ , hence  $\eta < p/2$ . Then,

$$\mathbb{E}\left[e^{\eta(X^{(\psi)} - Y^{(\psi)})}\right] \leq e^{8 \ln \psi} = e^{\frac{64p^2 \ln \psi}{8p^2}} = e^{\frac{v^2 p^2}{8}} \leq e^{\frac{v^2 \eta^2}{2}}. \quad (41)$$

By the tail bound shown in [19, Equation (2.18)] and noting that Equation (41) holds for  $\mathbb{E}\left[e^{\eta(Y^{(\psi)} - X^{(\psi)})}\right]$  by symmetry, we obtain,

$$\begin{aligned} \Pr\left[\left|\sum_{i=1}^{\tau} X^{(\psi)} - Y^{(\psi)}\right| \geq jr'\right] &\leq \begin{cases} 2\tau \exp\left(\frac{-\tau(jr')^2}{2\left(\frac{64 \ln \psi}{p^2}\right)}\right) & jr' \leq \frac{v^2}{b} \\ 2\tau \exp\left(\frac{-\tau jr'}{2\left(\frac{2}{p}\right)}\right) & jr' > \frac{v^2}{b} \end{cases} \\ &= \begin{cases} 2\tau \exp(-\tau(jr)^2) & jr' \leq \frac{v^2}{b} \\ 2\tau \exp(-2\tau jr \sqrt{2 \ln \psi}) & jr' > \frac{v^2}{b} \end{cases} \\ &\leq 2\tau \exp(-2\tau jr) \\ &= e^{-jr} 2\tau \exp(-(2\tau - 1)jr) \\ &\leq e^{-jr} 2\tau e^{-(2\tau - 1)jr} \leq e^{-jr}. \end{aligned}$$

□

**THEOREM 6.1.** Consider Algorithm 2 with fitness measure  $f^{\text{wrs}}$  and radius  $r \geq 1$  on BILINEAR. Define  $T = \min\{t \mid \text{OPT} \cap (P_t \times Q_t) \neq \emptyset\}$ . Let  $M_1 := |x_1 - \beta| + |y_1 - \alpha| > r$  be the initial Manhattan distance to the optimum. Then,  $\mathbb{E}[T] \leq \frac{M_1^2}{r} - r$ . If  $M_t \leq r$  then  $T = 1$ .

**PROOF OF THEOREM 6.1.** If  $M_t \leq r$  then the offspring of generation  $t$  will contain the optimum. Therefore, if  $M_1 < r$  then  $T = 1$ .

Let us assume  $M_t > r$ .  $M_t > r$  implies that either the optimum is part of  $P_1 \times Q_1$  or that  $P_1 \times Q_1$  is part of one or two contiguous quadrants of the search space. For simplicity we pessimistically assume that if  $M_1 > r$  holds then the optimum is not created and aim to show that the algorithm reaches  $M_t \leq r$ .

Given that  $P_1 \times Q_1$  is part of one or two contiguous quadrants of the search space, all conditions from Lemma 3.2 can be met except Conditions (17), (18) for  $x_{t+1}$ , and (27), (28) for  $y_{t+1}$ . In addition, Conditions (26) and (36) cannot be met for the algorithm considered because of how the offspring is created.

There are sixteen different combinations in which the conditions from Lemma 3.2 can be met whilst  $P_t \times Q_t$  is part of at most two quadrants. Thanks to the symmetry of the search space there are only three different behaviours (with respect to the change in distance to the optimum). We deal with each set of conditions (and behaviours) separately.

If Condition (19) is met for  $x_{t+1}$ , then either Condition (34) or (35) is met. Similarly, if Condition (29) is met for  $y_{t+1}$ , then either Condition (24) or (25) is met. These conditions represent the case where  $P_t \times Q_t$  is spread in exactly two quadrants, and the behaviour of the algorithm in all these cases is the same, therefore we only show the analysis for the former (Conditions (19) and (34)). For this case  $x_t, x_{t+1} \geq \beta$ , and the change in Manhattan depends on the starting position of  $y_t$  with respect to  $\alpha$ . If  $y_t \geq \alpha + 1$ , the change in Manhattan distance is

$$\begin{aligned} M_t - M_{t+1} &= |x_t - \beta| + |y_t - \alpha| - (|x_{t+1} - \beta| + |y_{t+1} - \alpha|) \\ &= x_t - \beta + y_t - \alpha - (x_{t+1} - \beta + \alpha - y_{t+1}) \\ &= x_t + y_t - 2\alpha - (x_t - r) + (y_t - r) \\ &= 2y_t - 2\alpha \geq 2. \end{aligned}$$

If  $y_t \leq \alpha$ , the change in Manhattan distance is

$$\begin{aligned} M_t - M_{t+1} &= |x_t - \beta| + |y_t - \alpha| - (|x_{t+1} - \beta| + |y_{t+1} - \alpha|) \\ &= x_t - \beta + \alpha - y_t - (x_{t+1} - \beta + \alpha - y_{t+1}) \\ &= x_t - y_t - (x_t - r) + (y_t - r) = 0. \end{aligned}$$



The second set of conditions that share the same behaviour is the conditions  $(\{(24),(25)\} \times \{(34),(35)\}) \cup \{(20) \text{ and } (34),(22) \text{ and } (35),(25) \text{ and } (30),(24) \text{ and } (32)\}$ . In the following we show the analysis of the case where (24) and (34) hold and the other cases follow by symmetry. The Conditions (24) and (34) imply that  $x_t, x_{t+1} > \beta$  and  $y_t, y_{t+1} > \alpha$ , therefore, the change in Manhattan distance is

$$\begin{aligned} M_t - M_{t+1} &= |x_t - \beta| + |y_t - \alpha| - (|x_{t+1} - \beta| + |y_{t+1} - \alpha|) \\ &= x_t - \beta + y_t - \alpha - (x_{t+1} - \beta + y_{t+1} - \alpha) \\ &= x_t - y_t - (x_t + r + y_t - r) = 0. \end{aligned}$$

The third and final set of conditions that share the same behaviour is  $\{(20) \text{ and } (35),(22) \text{ and } (34),(24) \text{ and } (30),(25) \text{ and } (32)\}$ . As before we show the analysis for the first case in the set and the others follow by symmetry. The Conditions (20) and (35) imply that  $x_t, x_{t+1} > \beta$ ,  $y_t > \alpha$  and  $y_{t+1} = \alpha$ . We note that the selection mechanism selects  $x_{t+1}$  uniformly at random when Condition (20) is met, therefore we compute the expected value of the change in Manhattan distance as follows.

$$\begin{aligned} E[M_t - M_{t+1} \mid (20) \wedge (35)] &= |x_t - \beta| + |y_t - \alpha| - (E[|x_{t+1} - \beta|] + |y_{t+1} - \alpha|) \\ &= x_t - \beta + y_t - \alpha - (E[x_{t+1}] - \beta + y_{t+1} - \alpha) \\ &= x_t - y_t - (E[x_{t+1}] + y_t - r) \\ &= x_t + r - E[x_{t+1}]. \end{aligned}$$

To compute  $E[x_{t+1}]$  we note that it is chosen from the values  $[x_t - r, x_t + r]$  and each value has the same probability of being chosen of  $\frac{1}{2r+1}$ , hence

$$\begin{aligned} E[M_t - M_{t+1} \mid (20) \wedge (35)] &= x_t + r - \left( x_t - r + \sum_{i=0}^{2r} \frac{i}{2r+1} \right) \\ &= x_t + r - \left( x_t - r + \frac{2r(2r+1)}{2(2r+1)} \right) = r. \end{aligned}$$

We note that in the worst case ( $x_{t+1} = x_t - r$  in the sum) the algorithm remains with the same distance, therefore we have shown that in all cases  $M_t - M_{t+1}$  is at least 0. Another thing to point out is that the algorithm only reduces  $M_t$  when  $(x_t, y_t)$  and  $(x_{t+1}, y_{t+1})$  are in different quadrants or when one condition in the third set of conditions is met (i. e.  $\{(20) \text{ and } (35),(22) \text{ and } (34),(24) \text{ and } (30),(25) \text{ and } (32)\}$ ). We call these events *crossing* an axis. Then, to compute the expected runtime we need to compute how many generations it takes to cross an axis and the number of crossings that are needed to reach the optimum.

The number of generations that takes to cross an axis from time  $t$  is at most  $\lceil M_t/r \rceil$ , because in every generation  $t$  that the algorithm does not cross an axis either  $x_{t+1}$  or  $y_{t+1}$  approaches  $\beta$  or  $\alpha$  by  $r$ . Since the algorithm does not increase  $M_t$ , then we can bound all the times between crossing the axes by  $\lceil M_1/r \rceil$ .

Now, we compute the number of crossings that are needed to reach the optimum. Let  $E^{(1)}$  denote the event that the algorithm crosses the axis at time  $t+1$ , let  $E^{(2)}$  denote the event that the algorithm crosses the axis at time  $t+1$  while  $P_t \times Q_t$  is spread in exactly two quadrants and let  $E^{(3)}$  denote the event that the algorithm crosses the axis at time  $t+1$  while one condition in the third set of conditions is met. Then,

$$\begin{aligned} E[M_t - M_{t+1} \mid E^{(1)}] &= \Pr[E^{(2)} \mid E^{(1)}] E[M_t - M_{t+1} \mid E^{(2)} \wedge E^{(1)}] \\ &\quad + \Pr[E^{(3)} \mid E^{(1)}] E[M_t - M_{t+1} \mid E^{(3)} \wedge E^{(1)}] \\ &\geq \Pr[E^{(2)} \mid E^{(1)}] 2 + \Pr[E^{(3)} \mid E^{(1)}] r \end{aligned}$$

Note that  $\Pr[E^{(2)} \mid E^{(1)}] + \Pr[E^{(3)} \mid E^{(1)}] = 1$  and  $r \geq 1$ . Therefore, the  $E[M_t - M_{t+1} \mid E^{(1)}] \geq 1$ . Let  $\tau$  be the number of crossings that are needed to reach a point with  $M_t \leq r$ . Using the Additive Drift Theorem [1] with a potential function  $g(M_t) = \min\{M_t - r, 0\}$  we can see that

$$E[\tau] \leq \frac{M_1 - r}{1}$$

We now compute the  $E[T]$  as follows,

$$E[T] \leq E[\tau] \left\lceil \frac{M_1}{r} \right\rceil \leq (M_1 - r) \left( \frac{M_1}{r} + 1 \right) = \frac{M_1^2 - r^2}{r}$$

□